THE VCG MECHANISM, THE CORE, AND ASSIGNMENT STAGES IN AUCTIONS*

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The Vickrey-Clarke-Groves (VCG) mechanism is one of the most compelling constructs in mechanism design, but complementarities create the possibility of non-core and even zero-revenue outcomes. In this article, we develop a theory of complementarities in one of the most natural applications for the VCG mechanism: assignment stages of spectrum auctions. When two opponents bid on distinct parts of a bidder's preferred assignment, they set up a *direct* bidder complementarity. In addition, the constraint in assignment stages that guarantees contiguous spectrum to each participant is shown to enable a new, *indirect* complementarity. As a result, non-core outcomes, together with all accompanying anomalies, must be expected in VCG assignment stages. We then examine a conspicuous recent spectrum auction, the FCC's Broadcast Incentive Auction, and we provide the first empirical documentation of zero-revenue outcomes. *JEL codes:* D44, D47

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I. Introduction

One of the most elegant and compelling constructs in the toolbox of mechanism design is the Vickrey-Clarke-Groves (VCG) mechanism.[1] In appropriate economic environments, asking agents to report their valuations for goods, allocating the goods efficiently relative to these reports, and compensating agents according to an opportunity cost calculation relative to these reports gives rise to an elegant mechanism. Truth-telling becomes a dominant strategy inducing efficiency.

However, almost two decades ago, economists added a cautionary amendment to our understanding: complementarities of goods can induce complementarities of agents, leading to potential anomalies in the VCG mechanism. The auction may result in uncompetitively low revenues (i.e., VCG payments that lie outside the core, in the sense that there exists a coalition of losing bidders who offered to pay more than the winning prices for some of the goods), and may invite perversities such as "loser collusion".[2]

A zero-revenue outcome, the most extreme ramification of this problem, becomes a theoretical possibility. Consider the "local-local-global" (LLG) model, with two goods (A and B) and three bidders (1, 2 and 3) possessing the following valuations:[3]

$$\begin{array}{llll} v_1(AB)=2 & v_1(A)=2 & v_1(B)=0 & v_1(\varnothing)=0\,; \\ v_2(AB)=2 & v_2(A)=0 & v_2(B)=2 & v_2(\varnothing)=0\,; \\ v_3(AB)=2 & v_3(A)=0 & v_3(B)=0 & v_3(\varnothing)=0\,. \end{array}$$

Local bidder 1 values only good A, at 2; and local bidder 2 values only good B, also at 2. The global bidder 3 views goods A and B as perfect complements and values only the AB combination at 2. Observe that the global bidder's complementary preferences for goods A and B make the two local bidders complementary in the auction. In the VCG mechanism, good A is awarded to bidder 1 and good B is awarded to bidder 2, since this is the value-maximizing allocation. However, the price paid by each of the local bidders is zero: the social surplus with both local bidders present equals 4, while the social surplus with either local bidder absent equals 2, implying that each local bidder should retain the entire social surplus that it creates. As such, the VCG mechanism's revenues of zero are disappointing and the outcome is outside the core, since there is a blocking coalition (the seller and the global bidder) whose coalitional value equals 2.

Low revenues by themselves are not necessarily a big concern for some sellers, especially government agencies that prioritize efficiency. At the same time, the possibility of zero-revenue outcomes allows bidders to obtain value without paying for it, almost a recipe for exploiting the auction mechanism.

Despite the theoretical literature, we are unaware of any documented empirical examples of zero-revenue outcomes. One possible reason for the dearth of known examples is that these observations spawned a literature on core-selecting mechanisms,[4] and real-world auctioneers, aware of this problem, frequently include core adjustments in their rules. An alternative explanation for the lack of examples is

[1] Vickrey (1961), Clarke (1971), Groves (1973).
[2] Ausubel and Milgrom (2002, 2006).
[3] Ausubel and Milgrom (2002) at p. 5.
[4] Day and Raghavan (2007), Day and Milgrom (2008), Day and Cramton (2012).

that zero-revenue outcomes rely upon an extreme and unrealistic complementarity, rendering it merely a theoretical curiosity and not a practical problem.

In this article, we develop the relevant theory of complementarities and we document the occurrence of zero-revenue and other non-core outcomes in what may seem to be an unlikely place: the assignment stage of a spectrum auction. Auctions of spectrum licenses that are close substitutes are often dichotomized into two successive phases. First, in the allocation stage, bidders bid for quantities of "generic" licenses (spectrum blocks that are not yet assigned to any specific frequency). Second, in the assignment stage, bidders compete for different assignments of specific frequencies that correspond to their generic winnings from the allocation stage.

At first glance, the VCG mechanism may appear to be the most natural methodology for an assignment stage. With the bulk of the auction value already determined in the first stage, VCG provides a fast (single round, sealed bid) and simple (dominant strategy) way to elicit bidders' preferences for specific frequencies and to establish the value-maximizing assignment. What could possibly go wrong?

The answer offered by this article is that bidder complementarities—and, hence, non-core outcomes—are endemic to assignment stages. As such, all pathologies possible in the VCG mechanism are especially likely to materialize in this setting.

It is initially easy to overlook that complementarities would even be present in assignment stages of spectrum auctions. The complementarities that one typically associates with the telecom industry relate to economies of scale within a geographic region or to synergies between regions. Once the allocation stage has concluded and the number of spectrum blocks won by each bidder in each region has been determined, any of these complementarities should have been resolved. Another typical source of complementarities is a bidder's preference for contiguous spectrum. For example, in an auction of four spectrum blocks that are identified (in ascending order of frequencies) as the A, B, C and D blocks and are otherwise equivalent, a winner of two blocks in the allocation stage would much prefer to be assigned the "AB" combination, the "BC" combination, or the "CD" combination. However, spectrum regulators are fully cognizant of this preference and impose a "contiguity restriction" assuring that all winners of multiple blocks receive adjacent blocks. One might naïvely hope that this restriction would treat all residual complementarities.

We show that this hope is misplaced. While imposing the contiguity restriction does mitigate certain complementarities among bidders, it simultaneously enables a novel type of bidder complementarity. More precisely, we demonstrate that contiguity can produce complementarities *directly* or *indirectly*. While the former type is derived from complementarities in goods (like in the earlier example from Ausubel and Milgrom (2002)), the latter type arises from the joint restriction on the outcome of the assignment stage (all winners must get contiguous assignments). To illustrate the difference, consider an assignment stage with four blocks (A, B, C and D) that are being assigned to a two-block winner and to two one-block winners. If the two-block winner especially values the "AB" assignment, then a one-block winner who demands "A" and a one-block winner who demands "B" are complementary, directly. (See Example 1 of Section III.) Meanwhile, if a one-block winner especially values the "A" assignment, then a two-block winner who demands "AB" and a one-block winner who demands "C" are also complementary, but indirectly. We describe this complementarity as "indirect" on account that assigning "A" and "C", respectively, to one-block winners does not cause a direct conflict. Nonetheless, it creates a conflict due to contiguity; once the "C" block is assigned to a one-block winner, "AB" becomes the only feasible (contiguous) assignment for the two-block winner, leaving "D" as the only feasible assignment for the other one-block winner. (See Example 2 of Section III.) In both scenarios, neither complementary bidder would be individually responsible for depriving the opposing bidder of its desired assignment, making non-core outcomes possible. Note that, by definition, indirect bidder complementarities are impossible in assignment stages without the contiguity restriction.

In general, a bidder's desire for contiguous spectrum—or the auctioneer's restriction of winnings to be contiguous—is fundamentally inconsistent with *assignment substitutes*, a variation on the well-known gross substitutes property[5] which we prove is the "robust" condition for guaranteeing core outcomes in assignment stages. When the allocation and the assignment are determined together within a one-stage process, the unavoidable complementarities arising from contiguity may be completely obscured by the much larger values for generic blocks. As a result, the VCG outcome of the one-stage process may lie in the core, despite these complementarities. By way of contrast, the same unavoidable assignment complementarities, once isolated within an assignment stage under the two-stage approach, may stand out in stark relief. It then turns out that non-core VCG outcomes and all related anomalies are not merely theoretical possibilities, but rather are systematic attributes of the assignment stage.

Imposing the contiguity restriction dramatically reduces the number of feasible assignment options for bidders. This reduction has two major implications. First, it becomes possible to explain any set of assignment bids with additive assignment values. Unfortunately, such well-behaved preferences by themselves do not imply core outcomes unless extra restrictions are imposed. Second, and independently from assignment preferences, the contiguity restriction eliminates many opportunities for bidder complementarities, and is shown to eliminate all of them in symmetric assignment stages (i.e., when each bidder entering the assignment stage has won the same number of blocks). Under full symmetry, the assignment stage with contiguity restriction is tantamount to the (well-behaved) auction in which bidders have unit demands.[6] For example, when six spectrum blocks are being assigned to three two-block winners, the only feasible assignments give "AB" to one bidder, "CD" to another bidder and "EF" to the remaining bidder, yielding full substitutability and the corresponding core guarantees.

However, in asymmetric assignment stages, non-core outcomes are generally unavoidable. We prove that in most scenarios with $n \geq 3$ bidders, any bidder with a single desired assignment can be forced out of its desired assignment by a coordinated increase in its rivals' bids. We refer to such scenarios as *cost-free predation* on account that, due to the nature of the direct and indirect complementarities, the coalition of $n-1$ opponents is able to obtain this outcome at zero cost to coalition members. Cost-free predation is another troubling anomaly of the VCG mechanism in settings with complementarities, which can be especially concerning in assignment stages.

Finally, we provide a case study in all of the aforementioned issues by examining the assignment stage of the most conspicuous, recent spectrum auction: the US Federal Communication Commission's Broadcast Incentive Auction of 2016–17. In particular, in this auction, three areas in the assignment stage exhibited zero-revenue outcomes and almost one-fifth of the assignment areas with at least three bidders yielded VCG outcomes that were outside the core. We propose an additional exercise for

[5] The gross substitutes condition was applied to tâtonnement processes by Arrow, Block, and Hurwicz (1959) and was introduced to auctions by Kelso and Crawford (1982).

[6] Demange, Gale, and Sotomayor (1986).

identifying assignment rounds that had the "potential" for non-core outcomes (in the sense that there existed bidder complementarities waiting to become relevant) and we find that it almost triples the incidence of anomalies (from 18% to 50%). Finally, we look for "winner collusion" and "loser collusion" opportunities for coalitions of bidders and we find them in 72% and 53%, respectively, of the assignment areas. As such, we provide the first empirical documentation of zero-revenue outcomes and related phenomena that arise from complementarities in the VCG mechanism.

The remainder of this article is organized as follows. Section II reviews the motivation for two-stage approaches to spectrum auctions and the use of assignment stages. Section III develops the logic that preferences for contiguity can produce direct or indirect bidder complementarities. Section IV introduces a stylized model of the two-stage auction approach. Section V shows that assignment substitutes is a sufficient and "almost necessary" condition for a well-behaved assignment stage, but it is inconsistent with contiguity concerns. Section VI studies the setting with enforced contiguity and establishes results on cost-free predation. Section VII provides an empirical look at these issues in the assignment stage of the Broadcast Incentive Auction. Section VIII shows that other feasibility constraints can have a similar effect as a contiguity requirement. Section IX concludes.

II. The Motivation for Assignment Stages in Spectrum Auctions

The critical stylized fact behind two-stage approaches to auctioning spectrum and the use of assignment stages is that bidders obtain value from receiving contiguous frequencies. In spectrum auctions of the past decade, the simplest justification for this fact has been in the LTE technology that is part of the 4G standard for mobile phones. Quite simply, a mobile provider can obtain significantly greater data throughput from two adjacent 10 MHz frequency blocks than from two noncontiguous 10 MHz blocks. However, the desire for contiguous spectrum goes well beyond LTE technology. The enhanced value of contiguity has been part of the landscape of spectrum valuation since the 1990s—and mobile providers continue to desire contiguous spectrum for deployment of the newest 5G technology.

Concurrently, spectrum regulators have exhibited a strong preference for allocating spectrum by auction rather than by administrative process, and for dynamic auction formats over sealed-bid auction formats. A wide range of reasons have been given for the latter preference, including: the price discovery induced by a dynamic auction;[7] the more aggressive bidding, arising from the lessened winner's curse, of a dynamic auction;[8] the greater transparency of a dynamic auction process;[9] the privacy preservation of winning bidders' values in a dynamic auction process;[10] and the cognitive simplicity of a dynamic auction over the corresponding sealed-bid auction.[11] Whether one accepts the reasons given or not, there is no mistaking that the vast preponderance of recent spectrum allocations worldwide utilize dynamic auction formats.

The desire for contiguous frequencies creates numerous possibilities for mischief in traditional dynamic auction formats. For example, the simultaneous multiple round auction (SMRA), an extremely popular

[7] Ausubel and Cramton (2004), Cramton (2013).
[8] Milgrom and Weber (1982).
[9] Ausubel (2004).
[10] Rothkopf, Teisberg, and Kahn (1990), Ausubel (2004).
[11] Kagel, Harstad, and Levin (1987).

format for the first 20 years of spectrum auctions (which is dynamic but just a single stage, in which bidders bid on specific frequencies), does not support contiguity restrictions. It enables anti-competitive strategies such as when, in one auction, "one bidder was attempting to assemble five adjoining licenses and the spoiler continued to bid on the middle license in a strategy that became known as 'giving the middle finger'."[12] When licenses are close substitutes, splitting the auction into two stages provides an elegant solution to this problem: limited to bids for generic blocks during the allocation stage and contiguous assignments in the assignment stage, bidders are unable to disrupt winnings of others by strategically positioning themselves within the band.

There are other practical disadvantages associated with traditional one-stage dynamic auction designs—with bidding on specific licenses—in settings with close substitutes. First, they frequently produce inordinately long auctions (with *m* licenses, it might take up to *m* rounds to increase the standing price on each license by just one price increment under the traditional SMRA format).[13] By contrast, two-stage approaches may admit a single price for the *m* licenses; then, only a single price needs to be incremented, requiring only a single round and drastically shortening the duration. Second, one-stage designs run a higher risk of inefficiency due to the fitting problem. Since bidders need simultaneously to discover both the "right" quantity allocation (constituting a larger portion of the auction value) and the "right" assignment (constituting a smaller portion of the auction value), a failure to coordinate on the assignment ("the fitting problem") might compromise discovering the "right" quantity of winnings and lead to significant welfare losses. By contrast, in the two-stage approach, the fitting problem never interferes with determining an optimal quantity allocation. Lastly, perhaps the strongest argument against the traditional approach is that the bidding language is not commensurate with the setting. When licenses are close substitutes, the need to select specific licenses becomes an artificial impediment that adds unnecessary complexity to the bidding decisions faced by bidders.

But there are also disadvantages of the two-stage approach. The most obvious one is a strategic problem faced by bidders when licenses are not perfectly substitutable. If specific frequency assignments have significant value differences, bidders are exposed to considerable uncertainty when bidding for generic blocks in the allocation stage since they are essentially bidding (and paying) for items of unknown value. In practice, auctioneers routinely engage in pre-auction product design. They organize licenses into distinct categories of generic blocks when, for example, some of the frequencies are encumbered, in order to mitigate this strategic problem and ensure that the "close substitutes" assumption is justified. A less obvious downside is the proclivity of assignment stages to bidder complementarities, which is at the heart of this article.

The first known instance of an assignment stage occurred in Trinidad and Tobago's GSM spectrum auction of June 2005. In the first, clock stage of the auction, up to two winners would be selected. In the second, assignment stage, the physical spectrum frequencies assigned to the winners would be determined.[14] Since then, assignment stages have become prevalent in spectrum auctions worldwide.

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[12] McAfee, McMillan, and Wilkie (2012), p. 181.

[13] The 2008 Canadian AWS auction extended 331 rounds and took 2 months. The 2014–15 US AWS-3 auction extended 441 rounds and took 2 ½ months.

[14] https://tatt.org.tt/AboutTATT/SpectrumManagement/FirstSpectrumAuction2005.aspx (accessed on March 16, 2021). The assignment stage of Trinidad and Tobago's auction utilized a first-price menu auction. Observe that, unlike VCG mechanisms, first-price menu auctions always generate core outcomes relative to the submitted bids. In the complete-information environments and with the refinement of coalition-proof Nash equilibrium considered

As of this writing, four of the five most recent US spectrum auctions allocating specific frequencies were two-stage auctions that included assignment stages. Similarly, the three most recent UK spectrum auctions, as well as the most recent Canadian and Australian spectrum auctions, have included assignment stages.

III. Bidder Complementarities in the Assignment Stage

In this section, we use simple examples to illustrate how bidder complementarities may arise in the assignment stage of a spectrum auction, in either of two ways. Consider a setting with four blocks (A, B, C and D) and three bidders each demanding at most two blocks, whose values are reported in Table 1. The efficient assignment consists of awarding block A to Bidder 1, block B to Bidder 2 and blocks C and D to Bidder 3.

Table 1: Example 1 — Direct Bidder Complementarity

	Bidder 1	*Bidder 2*	*Bidder 3*
Values*	$\underline{v}_1(1) = 150$ $v_1(A) = 170$ $\underline{v}_1(2) = 200$	$\underline{v}_2(1) = 100$ $v_2(B) = 120$ $\underline{v}_2(2) = 200$	$\underline{v}_3(1) = 300$ $\underline{v}_3(2) = 500$ $v_3(AB) = 520$
VCG Outcome	$(A,\ 100)$	$(B,\ 50)$	$(CD,\ 110)$

Two-Stage Approach

Efficient Allocation in Stage 1	1	1	2
Assignment Values in Stage 2	$\tilde{v}_1(A) = 20$	$\tilde{v}_2(B) = 20$	$\tilde{v}_3(AB) = 20$

* – for Bidder 1, the value for the A block is 170, the value for any other block (B, C or D) is 150, and the value for any 2 blocks is 200. The values for Bidders 2 and 3 are interpreted analogously.

First, consider allocating these blocks in a single stage using the VCG mechanism. With three bidders, it is straightforward to see that the only core constraints that may be violated by the VCG mechanism are those involving two bidders:

$$p_1^V + p_2^V = 150 \geq 20, \qquad p_1^V + p_3^V = 210 \geq 80, \qquad p_2^V + p_3^V = 160 \geq 30.$$

Note that, given the values, all three of these core constraints are easily satisfied. This is not surprising since bidders' values for blocks exhibit decreasing marginal returns under any assignment, ensuring that there are no complementarities among bidders in this example. As a result, all bidders are competing against each other and the VCG outcome is fully competitive (i.e., it lies in the core).

by Bernheim and Whinston (1986), the outcomes of first-price menu auctions are efficient. However, this result clearly does not extend to incomplete-information environments—and, if a first-price menu auction is used to resolve the LLG model described in Section I, the local bidders 1 and 2 face a severe free-rider problem when jointly competing against the global bidder 3.

Next, consider awarding these blocks using the two-stage approach. To avoid unnecessary details, we do not specify the mechanism used for the allocation stage. Instead, we simply assume that the allocation stage ended with the efficient allocation of generic blocks: Bidders 1 and 2 get one block each, and Bidder 3 gets two blocks.

By the rules of the assignment stage, all bidders are guaranteed to receive assignments corresponding to their winnings from the allocation stage. Then the only assignment value that Bidder 1 has in the assignment stage is an incremental value of 20 for obtaining block A. This value corresponds to its true marginal value of obtaining block A instead of any other block. Similarly, the assignment values of the other bidders are Bidder 2's incremental value of 20 for block B and Bidder 3's incremental value of 20 for the AB combination. Given these values, it is evident that Bidders 1 and 2 are no longer competing against each other. Instead, they are now complementary bidders who jointly compete against Bidder 3 in a classic LLG scenario in which Bidders 1 and 2 act as local bidders, and Bidder 3 acts as a global bidder. We refer to this scenario as a *direct* bidder complementarity since assigning either block A to Bidder 1 or block B to Bidder 2 physically conflicts with assigning AB to Bidder 3. With such a complementarity, if the auctioneer were to use the VCG mechanism for this assignment stage, Bidders 1 and 2 would have won their preferred assignments and paid zero—an obvious core violation given the positive bid of Bidder 3. Note that this example and the conclusion hold with or without the contiguity requirement on multi-block winnings, an important distinction emphasized in the next example.

Next, let us modify Example 1 with the values reported in Table 2. In particular, the only changes that are made in Example 2 (relative to Example 1) are an increase in value for the AB combination for Bidder 3 from 520 to 525, and a replacement of block B for block C in the value function of Bidder 2. The new efficient assignment consists of awarding block D to Bidder 1, block C to Bidder 2 and blocks A and B to Bidder 3. Similar to the previous example, it is easy to verify that the outcome of the one-stage VCG mechanism is still squarely in the core.

Table 2: Example 2 — Indirect Bidder Complementarity

	Bidder 1	*Bidder 2*	*Bidder 3*
Values*	$\underline{v}_1(1) = 150$	$\underline{v}_2(1) = 100$	$\underline{v}_3(1) = 300$
	$v_1(A) = 170$	$\boldsymbol{v_2(C) = 120}$	$\underline{v}_3(2) = 500$
	$\underline{v}_1(2) = 200$	$\underline{v}_2(2) = 200$	$\boldsymbol{v_3(AB) = 525}$
VCG Outcome	$(D,\ 80)$	$(C,\ 50)$	$(AB,\ 130)$

Two-Stage Approach

Efficient Allocation in Stage 1	1	1	2
Assignment Values in Stage 2	$\tilde{v}_1(A) = 20$	$\tilde{v}_2(C) = 20$	$\tilde{v}_3(AB) = 25$

* – for Bidder 1, the value for the A block is 170, the value for any other block (B, C or D) is 150, and the value for any 2 blocks is 200. The values for Bidders 2 and 3 are interpreted analogously.

Similar to Example 1, we analyze the assignment stage of the two-stage approach after the allocation stage that induced the efficient allocation of generic blocks. The residual assignment values for this assignment stage are reported in Table 2.

First, suppose that there is no contiguity restriction on multi-block winnings. In this case, Bidder 2's demand for block C does not overlap with other demands, implying that no bidder complementarities are present. If the auctioneer were to use the VCG mechanism for this assignment stage, Bidder 3 would have won the AB assignment and paid 20, the opportunity cost imposed by Bidder 1, while Bidder 2 would have received block C for free—a fully competitive outcome.

Alternatively, suppose that the auctioneer enforces the contiguity restriction that has been routinely imposed (for persuasive reasons, as recalled in Section II) in recent spectrum auctions. Since Bidder 3 is required to win two adjacent blocks, it is now impossible for Bidders 1 and 2 to win their desired blocks simultaneously. Given that Bidders 2 and 3 do not have a direct conflict, they are now complementary bidders who jointly compete against Bidder 1. This example is unusual because it is driven by the contiguity restriction rather than by a standard goods complementarity of the global bidder. Indeed, in this recast LLG scenario, Bidder 1 (a one-unit winner) has effectively taken the role of "global" bidder and Bidder 3 (the two-unit winner) has effectively become a "local" bidder. We refer to this scenario as an *indirect* bidder complementarity, since assigning block C to Bidder 2 does not physically conflict with assigning block A to Bidder 1 (but it does so indirectly through the contiguity restriction). If the auctioneer were to use the VCG mechanism for this assignment stage, Bidders 2 and 3 would have won their preferred assignments and paid zero—an obvious core violation given the positive bid of Bidder 1.

Both examples above illustrate how relatively small assignment complementarities can be "swamped" by much larger bidders' values for generic blocks in a one-stage formulation, ensuring that there are no complementarities among bidders and that the outcome of the one-stage VCG mechanism is in the core. By contrast, the same small complementarities, direct or indirect, are displayed in stark relief once the assignment is isolated from the allocation under the two-stage approach. In other words, dichotomizing a spectrum auction into two stages may turn irrelevant bidder complementarities into relevant ones.

IV. Model

An auctioneer seeks to auction m indivisible blocks of spectrum denoted by set $M = \{1, ..., m\}$. The set of all possible combinations of blocks (i.e., packages) is denoted by $2^M = \{ z \mid z \subseteq M\}$. The auctioneer allocates these blocks to a set of bidders $N = \{1, ..., n\}$. For each bidder $i \in N$, the bidder's preferences over packages are characterized by a value function $v_i(\cdot)$. The value of the null package, $\varnothing$, is normalized to zero. The assumptions made about value functions are as follows:

(A1) *Pure Private Values:* Each bidder knows its own value for any package and this value is not affected by the values of other bidders or the packages allocated to other bidders;

(A2) *Quasilinear Utility:* Each bidder i's payoff from winning package z and making a payment of p is given by $v_i(z) - p$.

The restrictions on possible winnings imposed by the auctioneer (if any) are embedded into a set of admissible packages $\Omega \subseteq 2^M$. For example, the auctioneer can limit the number of blocks that can be won by individual bidders or restrict multi-block winnings to form a contiguous swath of spectrum when blocks are adjacent to each other.[15]

When bidders view blocks in M as being sufficiently similar to each other (in other words, each bidder cares mostly about the quantity of blocks it wins than the specific blocks), the auctioneer can simplify the auction process by dichotomizing it into two stages. In the first (allocation) stage, all blocks in M are treated as a homogeneous good and bidders bid on quantities of generic blocks. The winning allocation in terms of generic blocks and associated payments are determined using some auction mechanism. The central subject of our article is the second (assignment) stage of the auction, while the exact auction mechanism used for the allocation stage is out of scope.

In the assignment stage, each first-stage winner submits bids for all admissible packages in Ω consisting of the number of blocks it won in the first stage. The winning assignments and associated assignment payments are determined by the standard rules of the VCG mechanism, where the interpretation of counterfactual opportunity costs is not that bidder i is absent from the assignment stage, but merely that bidder i bids zero for all of its admissible packages. Note that bidding in the assignment stage is optional since all winners of the allocation stage are necessarily assigned a quantity of specific blocks corresponding to their first-stage winnings.

Formally, let $\Omega(q)$ denote the set of all assignment options that are consistent with quantity q, i.e.,

$$\Omega(q) = \left\{ (z_1, \ldots, z_m) \in \Omega : \sum_{k=1}^{m} z_k = q \right\}, \tag{4.1}$$

and let $\underline{v}_i(q)$ denote the value of bidder i's least preferred assignment option corresponding to q generic blocks, i.e.,

$$\underline{v}_i(q) = \min_{z \in \Omega(q)} v_i(z). \tag{4.2}$$

Since bidder i is guaranteed some assignment in the second stage, its true marginal values for various assignments consistent with quantity q are given by:

$$\tilde{v}_i(z \mid q) = v_i(z) - \underline{v}_i(q) \qquad \forall z \in \Omega(q). \tag{4.3}$$

Let Y denote the set of all feasible quantity allocations of generic blocks, i.e.,

$$Y = \left\{ (y_1, \ldots, y_n) : y_i \in \{0, \ldots, m\} \quad \forall i \in N \quad \text{and} \quad \sum_{j \in N} y_j \le m \right\}. \tag{4.4}$$

[15] Note that the set of admissible packages Ω can be bidder-specific if needed.

For the rest of the article, we refer to an assignment stage for assigning profile of generic winnings $y \in Y$ as an assignment stage y. The set of all feasible assignments in the assignment stage y is given by:

$$X(y) = \left\{ (x_1,...,x_n): \; x_i \in \Omega(y_i) \quad \text{and} \quad x_i \cap x_j = \varnothing \;\; \forall i, j \in N: i \neq j \right\}, \tag{4.5}$$

and the coalitional value function for bidders in coalition $C \subseteq N$ on a subset of feasible assignments $X \subseteq X(y)$ is given by:

$$\tilde{w}_C(X) = \max_{x \in X} \sum_{j \in C} \tilde{v}_j(x_j \mid y_j) \; . \tag{4.6}$$

Note that definition (4.6) does not treat bidders outside coalition C as being absent but merely ignores their assignment values while still assigning them to feasible assignment options in set $X \subseteq X(y)$.

For assignment stage $y \in Y$, assignment $x = \left(x_1,...,x_n\right) \in X(y)$ is efficient if

$$\sum_{j \in N} \tilde{v}_j(x_j \mid y_j) = \tilde{w}_N(X(y)) \, . \tag{4.7}$$

Obviously, obtaining efficiency in the assignment stage does not imply the standard notion of allocative efficiency relative to value functions $\left\{v_i(\cdot)\right\}_{i=1}^{n}$ unless quantity profile y consists of efficient quantities of generic blocks.

A Vickrey outcome for assignment stage y consists of an efficient assignment $x = \left(x_1,...,x_n\right)$ and a payment vector $p^V = \left(p_1^V,...,p_n^V\right)$ where

$$p_i^V = \tilde{w}_{N\setminus\{i\}}(X(y)) - \sum_{N\setminus\{i\}} \tilde{v}_j(x_j \mid y_j), \tag{4.8}$$

and a core outcome consists of an efficient assignment $x = \left(x_1,...,x_n\right)$ and a payment vector $p^C = \left(p_1^C,...,p_n^C\right)$ belonging to the set of core payments $CP(x)$ given by:

$$CP(x) = \left\{ p^C \in \mathbb{R}_+^n : \; \tilde{w}_C(X(y)) - \sum_{C} \tilde{v}_j(x_j \mid y_j) \;\leq\; \sum_{N\setminus C} p_j^C \;\leq\; \sum_{N\setminus C} \tilde{v}_j(x_j \mid y_j) \quad \forall C \subseteq N \right\}. \tag{4.9}$$

Both the Vickrey outcome and core outcomes with respect to bids are defined in the analogous way by substituting bids for values.

We do not model the first (allocation) stage of the two stage approach in this article. Instead, we simply assume that the winning allocation of generic blocks is given by $y^* \in Y$. The second (assignment stage) is modeled as a Vickrey auction. Each bidder i submits bid $b_i(z) \geq 0$ for each assignment option $z \in \Omega(y_i^*)$. The winning assignment x^* is the solution to:

$$x^* \in X^* = \arg\max \sum_{j \in N} b_j(x_j) \qquad \text{s.t.} \quad x \in X(y^*). \tag{4.10}$$

If X^* contains multiple solutions, the auctioneer selects x^* according to a pre-determined tie-breaking procedure. The assignment payment p_i^* for each bidder i is calculated using Vickrey formula (4.8) by substituting values $\tilde{v}_i(z \mid y_i^*)$ with the corresponding bids $b_i(z)$ for each assignment option $z \in \Omega(y_i^*)$. The assignment payoff of bidder i (fully attributed to the assignment stage) is given by $\pi_i^* = \tilde{v}_i(x_i^* \mid y_i) - p_i^*$. Ultimately, after two stages, bidder i wins bundle x_i^* and pays the sum of two payments, one payment from the first stage for winning y_i^* generic blocks and assignment payment p_i^* from the second stage for obtaining x_i^*.

Note that all definitions above include all bidders in set N, even bidders who did not win any generic blocks in the allocation stage and, consequently, do not actively participate in the assignment stage. For such bidders, definitions above imply that $x_i^* = \varnothing$ and $p_i^* = 0$ whenever $y_i^* = 0$. As notation for the active bidders in assignment stage y, let $N(y) = \{\, i \in N : \ y_i > 0 \,\}$ and $n(y) = |N(y)|$, respectively, denote the set and the number of bidders who won at least one generic block in the allocation stage that led to assignment stage y.

The Vickrey payment rule in the assignment stage incentivizes bidders to bid truthfully on all feasible assignment options. For any given assignment stage $y = (y_1, \ldots, y_n)$, truthful bidding for bidder i consists of bidding zero for its least preferred assignment option (since the bidder is guaranteed some assignment) and bidding its true marginal value on any other available option, i.e.,

$$b_i(z) = \tilde{v}_i(z \mid y_i) = v_i(z) - \underline{v}_i(y_i) \qquad \forall z \in \Omega(y_i). \tag{4.11}$$

Due to well-known properties of the Vickrey auction, the following statements are true for any Vickrey assignment stage $y \in Y$:

1. Truthful bidding is a weakly dominant strategy for all bidders, implying the efficiency of winning assignment x^*; and
2. When $n(y) \leq 2$, the outcome of the assignment stage is always in the core with respect to values and bids.

Examples 1 and 2 from the previous section have already established our first finding:

Proposition 1. A set of value functions satisfying assumptions (A1) and (A2) and such that the VCG outcome of the (one-stage) VCG mechanism is in the core may nonetheless induce, under the two-stage approach, an assignment stage in which the VCG outcome lies outside the core with respect to assignment values.

Our main theoretical contributions are presented next. In Section V, we study properties of assignment stages without the contiguity restriction. Then, in Section VI, we consider assignment stages with the contiguity restriction, a standard way in which they are implemented in practice.

V. Assignment Substitutes

In this section, we assume that the auctioneer does not impose the contiguity restriction and the set of admissible packages is unconstrained (i.e., $\Omega = 2^M$). The well-known sufficient and "almost necessary" condition for the outcome of the VCG mechanism to be in the core is that bidders' preferences satisfy the gross substitutes condition.[16] In the following definition, let

$$x_i(p) = \arg\max_{z \in \Omega} \{ v_i(z) - p \cdot z \} \tag{5.1}$$

denote the demand correspondence of bidder i at price vector p, let P_i denote the set of nonnegative price vectors such that $x_i(\cdot)$ is single-valued, and let $x_{ik}(p)$ denote bidder *i*'s demand for the *k*th good at price $p \in P_i$.

Definition 1. Bidder i's value function $v_i(\cdot)$ satisfies *gross substitutes (GS)* if, for any two price vectors $p, p' \in P_i$ such that $p' \geq p$, $x_{ik}(p') \geq x_{ik}(p)$ for all $k \in M$ such that $p'_k = p_k$.

Since bidders' bids in the assignment stage are restricted to bundles with a constant number of blocks (corresponding to the generic winnings from the allocation stage), a weaker version of gross substitutes is sufficient for the assignment stage. In the following definition, let

$$x_i(p \mid q) = \arg\max_{z \in \Omega(q)} \{ v_i(z) - p \cdot z \} \tag{5.2}$$

denote the demand correspondence of bidder i restricted to bundles of q blocks at price vector p, let $P_i(q)$ denote the set of nonnegative price vectors such that $x_i(\cdot \mid q)$ is single-valued, and let $x_{ik}(p \mid q)$ denote bidder *i*'s restricted demand for the *k*th good at price $p \in P_i(q)$:

Definition 2. Bidder i's value function $v_i(\cdot)$ satisfies *assignment substitutes* for quantity q if, for any two price vectors $p, p' \in P_i(q)$ such that $p' \geq p$, $x_{ik}(p' \mid q) \geq x_{ik}(p \mid q)$ for all $k \in M$ such that $p'_k = p_k$.

First, we show that assignment substitutes for relevant quantities is a sufficient condition for the Vickrey outcome to be in the core in an assignment stage.

Proposition 2. Suppose that $\Omega = 2^M$. For an assignment stage $y = (y_1, ..., y_n)$, if the value function of each bidder $i \in N$ satisfies *assignment substitutes* for quantity y_i, then the Vickrey outcome is in the core with respect to the bidders' true assignment values.

Proof: For each bidder $i \in N(y)$, define the induced value function $v'_i(\cdot)$:

[16] Ausubel and Milgrom (2002).

$$v_i'(z) = \begin{cases} \tilde{v}_i(z \mid y_i), & \text{if } z \in \Omega(y_i), \\ -\infty, & \text{if } z \notin \Omega(y_i). \end{cases} \quad (5.3)$$

Since the value function $v_i(\cdot)$ satisfies assignment substitutes for quantity y_i, the induced value function $v_i'(\cdot)$ satisfies gross substitutes for each bidder $i \in N$. Consequently, the coalitional value function based on $v_i'(\cdot)$ is bidder-submodular (Ausubel and Milgrom, 2006, Theorem 8) and the Vickrey payoff vector of the assignment stage is in the core (Ausubel and Milgrom, 2006, Theorem 6). *QED*.

From Proposition 2, it immediately follows that a general condition for the Vickrey outcome of every assignment stage to be in the core is that all value functions $\{v_i(\cdot)\}_{i=1}^{n}$ satisfy *assignment substitutes* for each quantity $q = 1,...,m$.

Proposition 3. Gross substitutes implies assignment substitutes for each quantity $q = 1,...,m$, but the converse is not true.

Proof: The implication follows from Corollary 1.3 in Murota and Shioura (2018). For the converse, consider a counterexample with a bidder who (1) does not have any preferences for specific block assignments, but (2) has marginal values for generic blocks that violate decreasing marginal returns. *QED.*

To summarize, when bidders can bid for and win any packages (including non-contiguous packages) and their values satisfy gross substitutes, their induced values in the assignment stage satisfy assignment substitutes, and so the Vickrey outcome of the assignment stage must be in the core with respect to truthful bids. A violation of assignment substitutes does not automatically produce a non-core Vickrey outcome in the assignment stage. However, as one would expect from the literature on gross substitutes, if just one bidder has preferences violating assignment substitutes, it becomes possible to specify other bidders (with additive assignment preferences) such that the Vickrey outcome is not in the core with respect to the bidders' true values.

Next we analyze collusion possibilities in the assignment stage. In the VCG mechanism, individual bidders are incentivized to bid truthfully. However, a coalition of bidders can collude and profit from some deviations. A coalition of bidders can always weakly gain by eliminating opportunity costs imposed by members of the coalition on each other. To eliminate such costs, bidders can simply bid zero on each assignment option that they do not win (i.e., reducing losing bids). Sometimes a coalition can gain further by increasing its members' bids on assignment options that they win (i.e., raising winning bids). This might happen when coalition members are complementary. For any coalition, any coalitional gains obtained from reducing losing bids of coalition members will be referred to as *normal collusive gains*. Any additional gains obtained from raising their winning bids, after setting all losing bids to zero, will be referred to as *excess collusive gains*.[17] Note that reducing losing bids and raising winning bids can never cause a reduction in any bidder's payoff in the VCG mechanism.

[17] It is important to define excess collusive gains as a payoff increase after all losing bids are set to zero. When the losing bids are not set to zero, an increase in the winning bid can be equivalent to a reduction in losing bids, thus reflecting normal gains rather than excess gains.

In the following definitions, consider assignment stage y. Suppose that each bidder outside coalition C bids truthfully, i.e.,

$$b_i(z) = \tilde{v}_i(z \mid y_i) \qquad \forall z \in \Omega(y_i) \qquad \forall i \notin C. \tag{5.4}$$

and that each bidder in coalition C bids according to a *single-option* strategy

$$b_i(z) = \begin{cases} \alpha_i & z = x_i \\ 0 & z \neq x_i \end{cases} \qquad \forall i \in C. \tag{5.5}$$

Further suppose that the α_i's are set sufficiently high such that each bidder $i \in C$ wins the corresponding assignment option x_i. Note that coalition members must bid on non-overlapping assignments to satisfy this definition. The single-option strategy ensures that all normal collusive gains have already been captured and any additional gains that may be possible are excess gains. Now we define notions of *winner collusion* and *loser collusion*, which are both instances of excess collusive gains.

Definition 3. For assignment stage y and bidders in coalition C, a *winner collusion* opportunity exists if there exists a bidding profile satisfying (5.4)–(5.5) such that x^* is efficient and if, for some $i \in C$, the joint coalitional payoff is strictly increasing in α_i (i.e., the joint coalitional payoff goes up when one of the coalition members raises its winning bid).

Denote $\pi_C^d(x) = \sum_{i \in C} \pi_i^d(x)$ the highest coalitional payoff that coalition C can achieve in the assignment stage that yields assignment x. This payoff includes both any normal gains from reducing losing bids and any excess gains from raising winning bids.

Definition 4. For assignment stage y and bidders in coalition C, a *loser collusion* opportunity exists if there exists a bidding profile satisfying (5.4)–(5.5) such that x^* is inefficient and if, for some $i \in C$, the joint coalitional payoff is strictly increasing in α_i, such that individual and coalitional payoffs satisfy

$$\pi_i^d(x^*) \geq \pi_i^V \quad \forall i \in C \qquad \text{and} \qquad \pi_C^d(x^*) > \pi_C^d(x), \tag{5.6}$$

where x is an efficient assignment.

Remark: Our definition of loser collusion generalizes the definition of loser collusion in Ausubel and Milgrom (2006), who restrict attention to collusion by bidders who win null bundles under truthful bidding. In our setting, with bidders always winning one of their assignment options, this is equivalent to collusion by bidders who win their least preferred assignment options under truthful bidding. Note that such bidders never have winner collusion opportunities and their normal collusive gains are always zero. As a result, the second part of condition (5.6) is always satisfied whenever a coalition can generate excess gains by inducing an inefficient assignment. We refer to instances under the restricted definition as *pure loser collusion*.

Our next result establishes that a coalition of bidders in the assignment stage cannot obtain excess collusive gains when all bidders outside the coalition have well-behaved preferences.

Proposition 4. Suppose that $\Omega = 2^M$. For assignment stage $y=(y_1,...,y_n)$ and coalition $C \subset N$, if the value function of each bidder $i \in N \setminus C$ satisfies assignment substitutes for quantity y_i, then bidders in coalition C lack any opportunities for winner collusion or loser collusion.

Proof: First, the coalitional value function for bidders in $N \setminus C$ satisfies assignment substitutes and therefore must be submodular in assignments. Second, both winner collusion and loser collusion opportunities require excess collusive gains. For assignment $x = \left(x_1,...,x_n\right)$, when excess gains exist for coalition C, there must be bidder $i \in C$ whose Vickrey payment depends negatively on the choice of α_j. Let $C' \subset C \setminus i$ contain all bidders $j \in C$ such that bidder j does not win its assignment x_j when calculating $\tilde{w}_{N\setminus i}(X(y))$, and let $C'' = (C \setminus i) \setminus C'$. It follows that

$$\tilde{w}_{N\setminus i}(X(y)) = \sum_{C''} \alpha_k + \tilde{w}_{N\setminus C}(X_{C''}) > \sum_{C\setminus i} \alpha_k + \tilde{w}_{N\setminus C}(X_{C-i}),$$

where $X_C = \left\{ \hat{x} \in X(y):\ \hat{x}_j = x_j \quad \forall j \in C \right\}$ is a subset of feasible assignments $X(y)$ where all bidders in C are restricted to their winning assignments. At the same time, all α_j $(j \in C)$ are set high enough for bidders in C to win their assignments which implies:

$$\tilde{w}_N(X(y)) = \sum_{C} \alpha_k + \tilde{w}_{N\setminus C}(X_C) \geq \sum_{C''+i} \alpha_k + \tilde{w}_{N\setminus C}(X_{C''+i}).$$

But then

$$\tilde{w}_{N\setminus C}(X_{C-i}) - \tilde{w}_{N\setminus C}(X_C) < \tilde{w}_{N\setminus C}(X_{C''})) - \tilde{w}_{N\setminus C}(X_{C''+i}),$$

contradicting submodularity (in assignments) of the coalitional value function for bidders in $N \setminus C$. *QED.*

VI. Adjacent Blocks and Contiguous Assignments

In this section, we consider assignment stages in which blocks in M are physically adjacent to each other as in a typical band plan of a spectrum auction, and in which the auctioneer restricts any multi-block winnings to be contiguous assignments. Formally, we assume that all blocks are taken from a contiguous band of spectrum such that block 1 is the bottom end of the band, block m is the top end of the band, and any interior block j $(2 \leq j \leq m-1)$ is adjacent to blocks $j-1$ and $j+1$. All bidders in N are constrained to win contiguous assignments (i.e., bundles in which the assigned blocks form a connected set), which is captured by the following description of the set of feasible bundles:

$$\Omega = \left\{ (z_1,...,z_m) \in 2^M :\ z_k z_{k'} = 1 \rightarrow z_s = 1 \quad \forall s = k,...,k' \quad \forall k,k' = 1,...,m:\ k < k' \right\}. \quad (6.1)$$

Given our earlier definitions, it follows that any bidder i who has won $q \geq 2$ blocks in the allocation stage is guaranteed/restricted to win an assignment that consists of q adjacent blocks (a contiguous assignment). For ease of exposition, we limit attention in this section to assignment stages in which all generic blocks have been allocated to bidders in N (i.e., there are no unsold blocks).

We start with the simple observation that the contiguity restriction and the assignment substitutes property cannot coexist. To see this, consider a bidder who has won two generic blocks in the allocation stage (out of blocks A, B, C or D) and has an extra marginal value of 20 for any assignment that includes block A. Without the contiguity restriction, the bidder satisfies assignment substitutes. But, with the contiguity restriction, such bidder has value of 20 for the AB combination, and zero for the BC and CD assignments. If the bidder demands AB at an initial price vector, its demand will switch to CD when the price of B is made sufficiently high, causing demand to decline for block A—a clear violation of assignment substitutes. Moreover, observe that the contiguity restriction has turned bidder preferences from exhibiting assignment substitutes to exhibiting a perfect complementarity between blocks A and B, inducing the exact bid pattern used in the examples of Section 3.

Our next observation is that, under the contiguity restriction, the number of feasible assignments for a bidder becomes quite small. Consider a bidder who won $q \geq 2$ generic blocks in the allocation stage. Without the contiguity restriction, the bidder has $\frac{m!}{q!(m-q)!}$ possible assignments. However, with the contiguity restriction, the same bidder has at most $m-q+1$ feasible assignment options (depending on the winnings of other bidders). The starkest implication of the contiguity restriction and the corresponding reduction in feasible options is the ability to express any set of assignment values using simple value functions. For example, observe that, with m available blocks and only $m-q+1$ assignment values, there are always enough degrees of freedom to explain any assignment values with an additive value function in which a bidder attaches a value (positive or negative) to each block and the value of any assignment is given by the sum of values of the blocks in the assignment.[18] As a result, we can limit attention to additive assignment preferences without any loss of generality.

Therefore, with the contiguity restriction, additive assignment preferences cannot by themselves provide core guarantees.[19] As a result, we must consider further limitations on preferences. One possible restriction on additive valuations is to limit the number of non-zero block values to just one. When a bidder has a positive value for a single block (the *favored* block), the bidder values any contiguous assignment with the favored block at a constant premium to any contiguous assignment without the favored block. Alternatively, when a bidder has a negative value for a single block (the *poisoned* block), the bidder values any contiguous assignment without the poisoned block at a constant premium to any contiguous assignment with the poisoned block. For example, consider a bidder who won two generic blocks (out of A, B, C and D) and favors block B by 10. The corresponding assignment values are 10 for AB and BC, and 0 for CD. Meanwhile, if block B is instead poisoned by 10, the corresponding assignment values are 10 for CD, and 0 for AB and BC.

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[18] Consider a bidder who won two generic blocks (out of blocks A, B, C or D). In the assignment stage with the contiguity restriction, the bidder can specify at most three assignment values, for AB, BC, and CD combinations. Suppose that these assignment values are given by $\tilde{v}(AB\,|\,2)=0,\ \tilde{v}(BC\,|\,2)=4,\ \tilde{v}(CD\,|\,2)=2$. Then, one possible set of block values are {0, 0, 4, -2} for A, B, C and D, correspondingly. These values can be used to recover a value function (up to a constant) for all bundles with two blocks.

[19] By contrast, in the standard allocation problem without any contiguity restriction, additive preferences are sufficient to exclude the presence of any complementarities, guaranteeing that the VCG outcome is in the core.

The next proposition establishes that a single non-zero block value restriction does provide core guarantees, but only when the same block is either simultaneously favored by all bidders or simultaneously considered poisoned by all bidders. We have:

Proposition 5. Suppose that there are no unsold blocks, all bidders have additive assignment values with a single non-zero component and the contiguity restriction is enforced. If the same block is either simultaneously favored by all bidders or simultaneously viewed as poisoned by all bidders, then the VCG outcome of the assignment stage is in the core with respect to bidders' assignment values.

Proof: If a bidder cannot be assigned the favored (or poisoned) block due to the contiguity restriction, the truthful bids of this bidder have no impact on the VCG outcome of the assignment stage. As a result, we can assume without loss of generality that the favored (or poisoned) block may be assigned to any of the $n(y)$ bidders participating in the assignment stage, notwithstanding the contiguity restriction.

For a setting with a single favored block, the VCG assignment stage is equivalent to a standard second-price auction in which $n(y)$ bidders compete for a single item (an assignment with the favored block). The bidder with the highest value for the favored block (the winner) gets an assignment with the favored block and pays the second-highest value. It is well known that the outcome of the second-price auction is in the core. For a setting with a single poisoned block, the VCG assignment stage is equivalent to a standard uniform-price auction, where $n(y)$ bidders compete for $n(y)-1$ items (an assignment without the poisoned block). The bidder with the lowest disutility from the poisoned block (the loser) gets the poisoned block, while its bid sets the price for all other bidders with greater disutility (winners) who avoid assignments with the poisoned block. It is well known that the outcome of the uniform-price auction among bidders with unit demands is in the core. *QED*.

The sufficient condition in Proposition 5 might be empirically relevant. Sometimes, in assignment stages, a particular block in a band is encumbered by a prior user or receives interference from an adjacent spectrum band (a poisoned block). When this is due to an encumbrance or interference, it affects all bidders in the same way, and so the scenario falls under Proposition 5. Nevertheless, the condition is clearly very special and non-core VCG outcomes are generally unavoidable. For example, even when each bidder favors a single block but the favored block varies across bidders, non-core VCG outcomes become possible, as evidenced by the examples of Section III (both examples belong to this subclass).

Since the class of values in Proposition 5 is overly restrictive, we should also consider restrictions on the allocation of generic blocks entering the assignment stage, which is the other primitive of the assignment stage. We have already observed that enforcing contiguity dramatically reduces the number of possible assignments in the band. It is easy to see that this reduction can potentially eliminate some opportunities for bidder complementarities. To illustrate, consider a two-block winner who competes for the BC assignment (out of AB, BC or CD) and faces two one-block winners. Under the contiguity restriction, the bidder cannot compete against both of them simultaneously, since it is not possible for one-block winners to win B and C jointly, and force the two-block winner to AD. Thus, independent of bidders' assignment preferences, core guarantees might arise naturally in settings in which the contiguity restriction eliminates *all* opportunities for bidder complementarities. Our next proposition identifies such settings. We refer to assignment stage y in which all bidders in $N(y)$ have won the same number of generic blocks as *symmetric*, and otherwise *asymmetric*.

Proposition 6. When there are no unsold blocks and the contiguity restriction is enforced, the VCG outcome is in the core with respect to bidders' bids in any symmetric assignment stage.

Proof. Suppose that each bidder in $N(y)$ has won q blocks. With the contiguity restriction, the set of feasible assignment options for each bidder is fully captured by the bidder's *position* in the band (the first position corresponds to blocks $1,\ldots,q$, the second position corresponds to blocks $q+1,\ldots,2q$, etc.). With the same set of bidding options, the symmetric assignment problem is equivalent to assigning positions: each bidder has to be assigned one position from the set $\{1, 2,\ldots, n(y)\}$. In this formulation, all bidders have unit demands, trivially satisfying assignment substitutes and, by Proposition 2, ensuring that the VCG outcome is in the core. *QED*.

Given Proposition 6, we immediately conclude that a symmetric assignment stage with a contiguity restriction never creates opportunities for winner collusion or loser collusion. Conversely, when the assignment stage with a contiguity restriction is asymmetric, it is trivial to generate scenarios with core violations.

Proposition 7. Suppose that there are no unsold blocks and the contiguity restriction is enforced. For any asymmetric assignment stage with $n(y) \geq 3$, it is possible to specify additive assignment values such that the VCG outcome is not in the core.

Proof: Given the asymmetry, there must be a pair of bidders i and j such that $y_i < y_j$. Suppose that both i and j bid $\alpha > 0$ for their leftmost assignment option (blocks $1, \ldots, y_i$ for bidder i and $1, \ldots, y_j$ for bidder j) and zero on all other feasible options. Also suppose that bidder k ($k \neq i, j$) bids α for the assignment option corresponding to blocks $y_i+1, \ldots, y_i+y_k$ (which is feasible for bidder k) and zero otherwise, and that all other bidders (if any) bid zero for all of their options. The resulting VCG outcome is outside the core since bidders i and k win assignments $1, \ldots, y_i$ and $y_i+1, \ldots, y_i+y_k$, respectively, while paying zero (by the same logic as in Example 1 from Section III). Finally, by our observation above, any set of bids submitted in the assignment stage with the contiguity restriction can always be represented by an additive value function. *QED*.

Our findings so far indicate that assignment stages with the contiguity restriction, outside of two somewhat special conditions identified in Propositions 5 and 6, are prone to bidder complementarities and non-core VCG outcomes. Ironically, the desirability of contiguous assignments, one of the main motivations for the two-stage approach and for assignment stages in spectrum auctions, turns out to be fundamentally incompatible with core guarantees in the assignment stage.

With non-core outcomes being generally unavoidable, it is interesting to ask a follow-up question of whether a given bidder can avoid being exploited, i.e., avoid being the victim of a non-core outcome in which its opponents deny the bidder a desired assignment without paying for it. In order to construct a precise exercise, we consider a single bidder i (referred to as the *target bidder*), with generic winnings of y_i, who bids $\alpha > 0$ on a single assignment option $z \in \Omega(y_i)$ (referred to as the *target option*). For a given assignment round, we assess the degree to which the target bidder is vulnerable to collusion by all other bidders in $N(y)$. Specifically, we ask if the target bidder's opponents can deprive the target bidder of its target option while paying zero in the VCG mechanism, a phenomenon that we shall call *cost-free predation*.

In general, one should think about cost-free predation as a new item on the list of problematic issues of the VCG mechanism in the presence of bidder complementarities, alongside the familiar problems of core violations and winner/loser collusion. However, cost-free predation can be more relevant in assignment stages in the following sense. Spectrum auctions routinely use set-asides and other instruments to promote competition in downstream markets for cellphone service, integrating them into allocation stages but not assignment stages. As a result, denying the most valuable frequencies during the assignment stage may be the only tool left for industry incumbents who wish to disadvantage an entrant who has acquired generic licenses in the allocation stage. Predation can be especially relevant in auctions with multiple assignment stages, held for different service areas and conducted sequentially, such as the one considered in the next section.[20]

For completeness, we state our findings regarding cost-free predation with and without the contiguity restriction, and then follow with a discussion.

Proposition 8. (*Cost-free predation without a contiguity restriction*) Consider assignment stage $y=(y_1,...,y_n)$ with $n(y)\ \geq\ 3$ and no unsold blocks, and suppose that a contiguity restriction is not enforced. For any choice of a target bidder and its target option, there exists a group deviation by opposing bidders that deprives the target bidder of its target option for free, with one exception:

- The target bidder has won just one generic block.

Under the exception, the opposing bidders must pay the full opportunity cost of depriving the target bidder of its target option.

Proof: Given target bidder *i* with $y_i \geq 2$ and at least two opponents available to form a coalition, two coalition members can always bid on distinct parts of the target option, thus setting up a direct bidder complementarity that yields zero Vickrey prices for all coalition members. However, with $y_i = 1$, it is impossible to set up a direct bidder complementarity. As a result, for any bid profile that deprives the target bidder of its target option, bidder $j \in N(y)$ who gets the target option as part of its assignment will be assessed an opportunity cost of at least α. *QED.*

Remark: The sufficient condition in Proposition 8 is fully driven by our assumption that the target bidder i bids for a single assignment option, which violates assignment substitutes when $y_i \geq 2$. By

[20] There are a number of possible circumstances under which incumbent mobile operators might wish to deprive entrants of specific spectrum blocks or to force them into specific blocks, in order to impede competition. In one scenario, a dominant operator may envision that an entrant will enter into a spectrum-sharing arrangement with another cooperating firm. By depriving the entrant of frequencies adjacent to the holdings of the cooperating firm—most blatantly by obtaining those frequencies itself—the dominant operator may be able to interfere with the entrant's spectrum-sharing intentions. In another scenario, an entrant may have existing holdings at specific frequencies, which are harmonics of some of the frequencies in the auction—such that broadcasting on both sets of frequencies from the same cell towers would create interference. By relegating the entrant to those particular frequencies, the incumbents could make the entrant's acquisitions in the auction much less useful.

Proposition 4, a target bidder who won at least two generic blocks can always prevent cost-free predation in an assignment stage by submitting a set of bids satisfying assignment substitutes.

Proposition 9. (*Cost-free predation with a contiguity restriction*) Consider assignment stage $y=(y_1,...,y_n)$ with $n(y) \geq 3$ and no unsold blocks, and suppose that a contiguity restriction is enforced. For any choice of a target bidder and its target option, there exists a group deviation by other bidders that deprives the target bidder of its target option for free, with two exceptions:

- A symmetric assignment stage; or
- An asymmetric assignment stage with $n(y) = 3$ in which the two bidders other than the target bidder have won the same number of generic blocks and in which the target option corresponds to the "middle" position (i.e., the unique feasible assignment option for the target bidder that does not include blocks 1 or m).

Under either exception, coalition members pay the full opportunity cost of depriving the target bidder of its target option.

Proof: For symmetric assignment stages, depriving a bidder of the target option costs full opportunity costs, by Proposition 6. For asymmetric assignment stages, we consider two cases: assignment stages with three bidders ($n(y) = 3$) and assignment stages with four or more bidders ($n(y) \geq 4$).

$n(y) = 3$: Suppose that bidder i , who has won y_i blocks, is the target bidder. *End positions:* Given the asymmetry, there must be coalition member j such that either $y_j < y_i$ or $y_j > y_i$. To deprive bidder *i* of an end position, the coalition sets up a direct bidder complementarity when $y_j < y_i$ or an indirect bidder complementarity when $y_j > y_i$, either of which forces the target bidder to the opposite end. *Middle positions:* Denote the other coalition member as bidder k . If $y_k \neq y_j$, the target bidder has two distinct middle assignment options, and the coalition can deprive the target bidder of either one by setting up a direct bidder complementarity and forcing the target bidder to an end position. Finally, if $y_k = y_j$, the target bidder has only one middle option, which cannot be targeted by a direct or indirect bidder complementarity. In this case, the coalition pays the full opportunity costs for forcing the target bidder to an end position.

$n(y) \geq 4$: For end positions, the argument is the same as above. For middle positions, we will show, using the logic of the $n(y) = 3$ case, that two of the coalition members can always set up a bidder complementarity in a sub-band that includes the target option. Any feasible target option must be consistent with some assignment $B_1B_2...B_{l-1}TB_{l+1}...B_{N(y)-1}$, where coalition members $B_1, B_2,..., B_{l-1}$ are assigned lower blocks than the target bidder, and coalition members $B_{l+1}, B_{l+2},..., B_{N(y)-1}$ are assigned higher blocks. Note that the target option is also consistent with all alternative assignments in which $B_1, B_2,..., B_{l-1}$ are assigned lower blocks than the target bidder (but in a different order) and in which $B_{l+1}, B_{l+2},..., B_{N(y)-1}$ are assigned higher blocks than the target bidder (but in a different order). Since $n(y) \geq 4$, at least one three-bidder sub-band exists—either $B_{l-2}B_{l-1}T$ or $TB_{l+1}B_{l+2}$ —in which the target option occupies the end position of the sub-band. If one of them is asymmetric, the logic of the $n(y) = 3$ case applies exactly. If these sub-bands are symmetric, there must be bidder j such that $y_j \neq y_i$ and an alternative assignment that is also consistent with the target option in which bidder j is situated adjacent to the target bidder (directly below or above), resulting in an asymmetric three-bidder sub-band with the target option in the end position. The only exception is an assignment in which bidder j gets the end position and the target bidder is adjacent to bidder j, e.g., $B_j T B_2 ... B_{N(y)-1}$, while all other coalition members but bidder j have won the same number of generic blocks as the target bidder. Then, the logic of the $n(y) = 3$ case can be applied to the middle position of the $B_j T B_2$ sub-band since bidder j has won a different number of generic blocks than bidder B_2. *QED.*

Remark: In the setting of Proposition 9, if the target bidder places positive bids for multiple assignment options, it can be shown, using the same arguments and subject to the same exceptions, that the coalition members are able to deny the target bidder its most preferred assignment without being charged the full opportunity cost of the denial.

As has been shown above, in assignment stages without the contiguity restriction, a bidder who won a single block is always protected against being a victim of non-core outcome while multi-block winners can protect themselves by bidding with substitutes preferences. In sharp contrast, in assignment stages with the contiguity restriction, a bidder is completely unprotected unless the assignment stage is fully symmetric ("no one is safe unless everyone is safe").

Our findings in this section show that the contiguity restriction has an ambiguous effect on the likelihood of non-core outcomes. On the one hand, the contiguity restriction removes any chance of substitutes preferences for multi-unit winners, enabling new complementarities among bidders. On the other hand, the dramatic reduction in feasible assignments eliminates some bidder complementarities, including a complete elimination in symmetric assignment stages.

In the next section, we provide an empirical case study of bidder complementarities, non-core outcomes and all related issues in the assignment stage of the FCC Broadcast Incentive Auction of 2016-17.

VII. The Assignment Stage of the FCC's Broadcast Incentive Auction

The FCC's Broadcast Incentive Auction included two separate but interdependent auctions: a reverse auction, where television broadcasters bid to relinquish their spectrum rights; and a forward auction, where wireless operators bid to acquire licenses for the freed-up spectrum.[21] For the forward auction, the FCC adopted a two-stage approach. In the first stage, bidders bid in a uniform-price ascending clock auction for generic blocks of spectrum in each of 416 distinct partial economic areas (PEAs). In the second stage, bidders bid for physical frequency assignments of their generic winnings.

[21] A detailed description of the Broadcast Incentive Auction can be found in Ausubel, Aperjis and Baranov (2017). All publicly-available bidding data can be found at https://auctiondata.fcc.gov/public/projects/1000.

The assignment stage was conducted as a sequence of sealed-bid auction rounds, organized in up to six parallel sessions. Bidders bid for their assignments independently in each region, in descending order of population. To accelerate the process and to enhance geographic contiguity, PEAs with the same winners and same winnings were grouped together into assignment areas.[22] As a result, the total number of assignment rounds was reduced from 416 to 228. Participating bidders were informed about their own assignments and payments in each round before they bid for assignments in the next round.

For the assignment stage, the FCC adopted the VCG mechanism with a contiguity requirement: each winner of generic blocks was guaranteed a contiguous assignment of spectrum within the assignment area. Bidders were invited to bid on all possible contiguous assignments corresponding to their generic winnings—even though some of these assignments would be incompatible with maintaining contiguity for other winners—in order to avoid disclosing information about the winnings of other bidders.

The bidding data from the assignment stage of the Incentive Auction presents us with a unique empirical opportunity to evaluate the extent of non-core outcomes in the VCG mechanism. We conduct five empirical exercises to identify actual and potential non-core outcomes, to find opportunities for winner and loser collusion, and to classify the nature of underlying bidder complementarities. For the exercises identifying winner and loser collusion (Exercises 4 and 5), we treat bidders' bids as their true values. On the one hand, this is an easy assumption to make, given the dominant strategy property of the VCG mechanism for individual bidders. On the other hand, the theory developed in the previous section shows that groups of bidders can sometimes benefit from misreporting their values. Moreover, such group deviations can be lucrative within the sequential assignment stage process of the Incentive Auction, where the same bidders bid repeatedly against each other. Thus, the assumption of truthful bidding is a limitation of our approach.

Before presenting the results of our exercises, let us provide a general sense of the relative block values expressed in the submitted bids of the assignment stage. To limit interpretational issues, we initially look only at bids for single-block assignment options, i.e., bids by winners of one generic block; note that 2,884 bids (60% of the 4,753 assignment stage bids) were single-block bids.[23] In Table 3, we report the average bid amount for each single-block assignment option, normalized by the number of MHz times the population of assignment regions—the standard price measure used in spectrum valuation. The average bid amounts ranged from 0.11 cents (per MHz-pop) for the A block to 3.94 cents for the F block. By comparison, the average price paid for generic spectrum in the allocation stage was about $1 per MHz-pop, almost two orders of magnitude higher than the assignment values. Interestingly, while average bids for blocks B – G were in a narrow range of 2.68 to 3.94 cents (per MHz-pop), the average bid for the A block was only 0.11 cents, indicating that bidders generally viewed the A blocks as the least desirable blocks. The reason for this was probably that block A lay adjacent to the guard band for TV channel 37, which is reserved for radio astronomy. As such, bidders knew that there would not be mobile spectrum available immediately adjacent to and below the A blocks—and they may have also had concerns that power restrictions would ultimately be placed on A blocks near radio observatories.

[22] When two or more PEAs had the same winners and the same generic winnings, they were combined into a single assignment area. In the assignment stage, a bidder would then bid for (and win) the same assignment option in all PEAs included in the assignment area.

[23] The restriction to single-block bids also has the effect of removing T-Mobile's bids from Table 3. T-Mobile won three or four blocks in most regions and bid so as to concentrate its holdings in the B, C and D blocks, so analyzing T-Mobile's bids would have little effect on the conclusions drawn in this paragraph.

Table 3: Average Bids Amount for Single-Block Assignment Options

	A	*B*	*C*	*D*	*E*	*F*	*G*
*Average Bid (cents/MHz*pop)*	0.11	3.73	2.91	3.11	3.71	3.94	2.68

Our first exercise is very straightforward and yet rather remarkable. In the assignment stage of the Incentive Auction, there were 18 assignment areas (out of 228) that generated zero revenues. In 15 of these instances, all bidders were able to obtain their most-preferred feasible assignments, confirming that these zero revenue outcomes were fully competitive. But most notably, in each of the remaining three instances of zero revenues, there was a single bidder (the same one in all three instances) who failed to win its most preferred assignment, only to see its opponents (also the same ones in all three instances[24]) get their preferred assignments for free, literally reenacting our Proposition 9 on cost-free predation. To the best of our knowledge, this is the first time that (non-core) zero revenue outcomes of the VCG mechanism have been documented in the field.

We now provide details for one of these instances. The assignment of seven blocks in PEAs 224 and 287 (DeKalb, IL and Kenosha, WI, which had been grouped together) was made in assignment round 37 (REAG 3). In this assignment area, Dish Network won one generic block, while T-Mobile and U.S. Cellular each won three generic blocks. All bids, including discarded incompatible bids, submitted by bidders in this round are reported in Table 4, with winning bids displayed in bold.

Table 4: Assignment Bids for PEAs 224 and 287 in $ million

Band Plan

A	B	C	D	E	F	G

Bidders (generic winnings)	Dish Network (1 block)	T-Mobile (3 blocks)	U.S. Cellular (3 blocks)
Compatible Bids (block(s), bid)	**A, 0**	ABC, 0.4m	ABC, 0
	D, 0	**BCD, 3.5m**	BCD, 0
	G, 0.237m	DEF, 0.010m	DEF, 4m
		EFG, 0	**EFG, 8.7m**
Incompatible Bids (block(s), bid)	B, 0.308m	CDE, 2.1m	CDE, 0.010m
	C, 0		
	E, 0		
	F, 0.310m		
VCG payment	**0**	**0**	**0**

The mechanics of the VCG payment calculations for this example are as follows. U.S. Cellular's bid of $8.7 million for blocks EFG prevented Dish Network from winning block G. Independently, T-Mobile's bid of $3.5 million for blocks BCD also prevented Dish Network from winning block G, since winning it would have forced U.S. Cellular to a noncontiguous assignment. Therefore, taking away the bids of either

[24] It is especially striking that each of these three instances involved exactly the same bidders, given that the identities of opponents in the assignment stage were kept anonymous until after the auction concluded.

T-Mobile or U.S. Cellular (but not both), Dish Network would still have been assigned block A. As a result, neither T-Mobile nor U.S. Cellular "caused" the assignment and so the VCG payments of both these bidders equaled zero, a non-core outcome resulting from an indirect bidder complementarity between the two bidders. Their joint payment would have needed to increase by $237,238 in order to be in the core. Furthermore, note that the outcome of this assignment round would have stayed the same even if Dish Network had bid substantially higher. In fact, the VCG payments for both T-Mobile and U.S. Cellular would have remained zero if Dish Network had bid as much as $3.1 million for block G.

In the two other assignment areas that generated uncompetitive zero-revenue outcomes, the VCG mechanics played out in exactly the same way.[25] In both assignment areas, Dish Network won one generic block, while T-Mobile and U.S. Cellular collectively won 6 generic blocks; and in both areas, an indirect bidder complementarity between two opponents prevented Dish Network from acquiring block G, creating core violations of $101,030 and $147,602, respectively. At the end of the day, an entrant was relegated to what we have seen to be the least desirable block (A), at zero cost to the two incumbents.

Our second exercise is to determine which assignment areas had VCG outcomes lying outside the core. We find 38 such instances (including the three zero-revenue instances identified above). The cumulative revenue shortfall across these 38 assignment areas was $4,411,699. That is, if the FCC had instead used a core-selecting payment rule—and assuming the same bids—the assignment stage gross revenues would have been $140,342,331 rather than $135,930,632 (about 3.25% higher). However, given that core-selecting auctions are not in general incentive compatible (i.e., bidders may gain from non-truthful bidding), the interpretation of this comparison is limited.

We then perform three additional exercises to identify other assignment areas in which the bids had the "potential" for non-core outcomes—or presented bidder coalitions with opportunities for egregious manipulations utilizing bidder complementarities. To motivate our third exercise, recall that the presence of complementarities is necessary, but *not* sufficient, to generate core violations. When all bidders have substitutes preferences, changing the magnitude of bidders' relative preferences by rescaling their bids would never produce a bidder complementarity, and consequently would never trigger a core violation. However, when valuations already contain complementarities, rescaling the valuations may cause complementarities that were irrelevant at the optimal allocation to become relevant. Therefore, rescaling bids is an easy way to show that an assignment area had the "potential" for non-core outcomes (without artificially introducing any new bidder complementarities). Our third exercise looks for assignment areas where rescaling the actual bids gives rise to core violations.

Our fourth exercise looks for coalitions with winner collusion opportunities, as defined in Section 5. To eliminate any normal collusive gains, we first reduce all losing bids to zero for all members of a coalition. We then set coalition members' winning bids to the minimum levels required for winning and check whether the members' payoffs increase when they simultaneously raise their winning bids from this point. Finally, our fifth exercise looks for coalitions with loser collusion opportunities, as defined in Section 5. We determine whether coalition members can increase their payoffs even more, compared to their optimal payoff under the fourth exercise, by strategically raising some of their losing bids and thereby altering the winning assignment.

[25] The assignment of PEAs 205 and 213 (Douglas City, CA and Bend, OR) occurred in assignment round 29 (REAG 6); and the assignment of PEAs 119, 206 and 297 (Yakima, WA; Wenatchee, WA and Pendleton, OR) occurred in assignment round 37 (REAG 6).

These three exercises can all be illustrated using the bidding data from assignment round 21, in which seven blocks were assigned in PEA 38 (Milwaukee, WI) and the VCG outcome was in the core. All bids submitted by bidders in this round are reported in Table 5, with the winning bids displayed in bold.

Table 5: Assignment Bids for PEA 38 (Milwaukee, WI) in $

Band Plan

A	B	C	D	E	F	G

Bidders (generic winnings)	Dish Network (1 block)	T-Mobile (3 blocks)	U.S. Cellular (2 blocks)	New Level (1 block)
Compatible Bids (block(s), bid)	A, 0	ABC, 0	AB, 10.3k	**A, 0**
	B, 0	**BCD, 20.2m**	BC, 0	B, 176.471k
	C, 0	CDE, 13m	CD, 50.2k	C, 176.471k
	D, 0	DEF, 1m	DE, 50.2k	D, 176.471k
	E, 0	EFG, 0.1m	**EF, 0**	E, 176.471k
	F, 1019.752k		FG, 10.3k	F, 176.471k
	G, 250.123k			G, 0
VCG payment	**186.771k**	**819.830k**	**0**	**0**

To illustrate our rescaling exercise, observe that given T-Mobile's very high bid for the BCD assignment, the rest of the assignment is determined by comparing Dish Network's bid for block G, on the one hand, with the sum of New Level's and U.S. Cellular's bids for the EFG blocks, on the other. This is tantamount to the "LLG" model with indirect bidder complementarities, where Dish Network takes the role of the global bidder, while New Level and U.S. Cellular take the role of local bidders. With the original bids, the global bidder prevails over the two local bidders, since 250.123k > 186.771k = 176.471k + 10.3k, and so the VCG outcome lies within the core. However, had the global bidder submitted sufficiently smaller bids—or had the local bidders submitted sufficiently larger bids—a core violation would have been triggered instead. Rescaling the bids and recomputing the VCG outcomes would detect such a scenario.

To illustrate our winner collusion exercise, consider a coalition consisting of Dish Network and U.S. Cellular. The opportunity costs of Dish Network winning block G come from U.S. Cellular (a cost of $10,300 from winning EF instead of FG) and New Level (a cost of $176,471 from winning A instead of E). Obviously, U.S. Cellular can eliminate its $10,300 in opportunity costs imposed on Dish by bidding zero on FG ("normal" collusive gains). Now suppose that both Dish Network and U.S. Cellular further increase their winning bids. Observe that Dish Network's bid for block G and U.S. Cellular's bid for EF create an indirect bidder complementarity, since each one by itself prevents New Level from winning block E. As a result, Dish Network's opportunity costs are directly reduced by the amount of U.S. Cellular's winning bid. By raising its bid on EF, U.S. Cellular can reduce Dish Network's VCG payment all the way to zero, for a total coalitional gain of $176,471 ("excess" collusive gains). Our winner collusion exercise considers which coalitions are able to extract such excess collusive gains.

Finally, to illustrate our loser collusion exercise, continue considering the coalition consisting of Dish Network and U.S. Cellular and suppose that, instead of the winner-collusion deviations, Dish Network raises its bid for F and U.S. Cellular raises its bid for AB (both losing options). With sufficiently high bids, the coalition will be successful in shifting T-Mobile from the BCD assignment to the CDE assignment.

Observe that this manipulation imposes \$7.2m in opportunity costs due to T-Mobile. However, since both bids conflict with T-Mobile's bid for BCD, creating an indirect bidder complementarity, neither Dish Network nor U.S. Cellular is individually responsible for displacing T-Mobile. As a result, the coalition gains \$790,230 (\$779,930 for Dish Network and \$10,300 for U.S. Cellular) from loser collusion, compared to only \$186,771 (\$176,471 + \$10,300) from the winner collusion manipulation described above. Most importantly, observe that this manipulation causes a value loss of \$6.42m to this market—the substantial collateral damage caused by loser collusion.

For an example of "pure" loser collusion, consider the coalition consisting of U.S. Cellular and New Level, each winning its least desirable option and obtaining a zero payoff. If their bids for options FG and E, respectively, were made substantially higher, they would have outbid Dish Network's bid for G and, due to their indirect bidder complementarity, would have paid zero and gained \$186,771 in total. Our loser collusion exercise determines which coalitions are able to extract additional collusive gains by altering the winning assignment.

Exercises 4 and 5 treat bidders' bids as their true values. The reader may observe that the group deviations required for winner and loser collusion are relatively subtle. At the same time, the deviations may be viewed as individually risky, in that each coalition member raises its own bid and risks paying more only to improve another member's payoff. As such, the assumption that bids correspond to true values may be taken as satisfactory for these exercises.

To present our empirical findings, we first provide short methodological descriptions of each exercise:

- Exercise 1: For each assignment area with zero revenues, does the VCG outcome calculated from the actual bids lie outside of the core?
- Exercise 2: For each assignment area, does the VCG outcome calculated from the actual bids lie outside of the core?
- Exercise 3: For each assignment area, does the VCG outcome calculated from any rescaled version of the bids lie outside of the core? To address this, we apply all possible combinations of scaling factors chosen from the nine-element set $\{\varepsilon + 0.25k : k = 0, 1, \ldots, 8\}$ independently to the bids of all of the bidders (i.e., we rescale each bid in the range from essentially 0% to 200% in 25% increments), and we recalculate the VCG outcome for each such combination.[26]
- Exercise 4: For each assignment area, is there a bidder coalition such that winner collusion is profitable? We consider all possible coalitions of size two and above (excluding the grand coalition)—and we check whether coalition members can increase their payoffs by changing their winning bids (after setting their losing bids to zero).
- Exercise 5: For each assignment area, is there a bidder coalition such that loser collusion is profitable? We consider all possible coalitions of size two and above (excluding the grand coalition)—and we check whether coalition members can increase their payoffs even more, compared to their optimal payoffs under the fourth exercise, by strategically raising some of their losing bids. For "pure" loser collusion, we only consider coalitions with zero VCG payoffs.

In addition, for each such instance identified, we classify the underlying bidder complementarity that causes the non-core outcome or that is being exploited by a colluding coalition. "Direct" and "Indirect" instances correspond to direct and indirect bidder complementarities, and "Mixed" category contains

[26] Note that our resulting estimates are conservative, since the grid of possible scaling factors considered is coarse.

instances in which there are multiple core violations, some of them caused by direct bidder complementarities and some of them by indirect complementarities.

Our empirical results are summarized in Table 6. Note that we can exclude 21 of the 228 assignment areas from consideration, as it is never possible to observe a non-core VCG outcome or an affirmative result to any of our exercises, in an auction with only two bidders.[27] Thus, we treat the sample size as being the 207 assignment areas with three or more bidders.

Table 6: Summary of Empirical Exercises for Incentive Auction Bidding Data

	Exercise 1	*Exercise 2*	*Exercise 3*	*Exercise 4*	*Exercise 5*	
	Zero-revenue non-core outcomes	*Non-core Outcomes*	*Rescaling Values*	*Winner Collusion*	*Loser Collusion*	*Pure Loser Collusion*
Number of Markets	18	207	207	207	207	55
Identified Markets	3	38	103	149	110	37
Number of Coalitions	–	–	–	1626	1626	55
Identified Coalitions	–	–	–	349	170	37
Classification of Underlying Bidder Complementarities						
Direct	0	7 (18.5%)	–	58 (17%)	67 (39%)	7 (19%)
Indirect	3 (100%)	29 (76.5%)	–	291 (83%)	103 (61%)	30 (81%)
Mixed	0	2 (5%)	–	–	–	–

Starting from the 38 original assignment areas with non-core outcome, the rescaling exercise finds an additional 65 areas with bidder complementarities that could have potentially caused non-core outcomes. The winner collusion exercise identifies 149 assignment areas in which coalitions could have exploited bidder complementarities for excess gains. Lastly, the loser collusion exercise finds 110 areas in which coalitions could have gained even more (compared to their winner collusion opportunities) by forcing a different assignment. In particular, for pure loser collusion, we consider all coalitions with zero VCG payoffs (there are 55 of them in total, each one in a different assignment area) and we find that in 37 of them losers can gain by altering the assignment. Cumulatively, these exercises reveal 134 additional assignment areas with bidder complementarities, bringing the total number of areas with potential issues to 172 (out of 207). Our classification of underlying complementarities further reveals that indirect bidder complementarities—the ones arising from enforced contiguity—appear to play a dominant role in these results.

These empirical results must be taken with some caution since the Incentive Auction data is limited to markets in which the number of generic blocks is seven, a prime number that rules out any possibility of symmetries that could have reduced the number of problematic instances. Nevertheless, these results suggest that the presence of various bidder complementarities, both direct and indirect, is a systematic feature of asymmetric assignment stages rather than something rare and exotic.

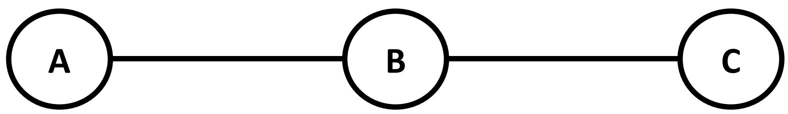
[27] With two bidders, it is well known that the VCG outcome lies within the core for all possible valuations. Similarly, it is easy to see that profitable winner collusion or loser collusion cannot occur in auctions with only two bidders.

VIII. Complementarities Arising from Other Feasibility Constraints

In the previous sections, we have found that a key source of the intensified bidder complementarities that arise in assignment stages is the contiguity requirement typically imposed by spectrum regulators. The purpose of the current section is to show that a contiguity requirement is not the only type of feasibility constraint that creates or amplifies bidder complementarities—in principle, any complex feasibility constraint can have a similar effect.

To illustrate, consider the scenario depicted in Figure 1. An auctioneer wishes to allocate three possible items (A, B and C), subject to the following restrictions: allocating A conflicts with allocating B; allocating B conflicts with allocating C; but allocating A does not conflict with allocating C. As such, conflicts among allocating these items are *not* transitive. Moreover, bidders seeking non-conflicting pairs of items prove to be complementary.

Figure 1: Non-Transitive Conflicts



Conflicts having this character are not uncommon. One example occurs in the same setting as the reverse auction of the FCC's Broadcast Incentive Auction. Three broadcasters seek television licenses for the same channel at locations approximately along a straight line. The broadcast locations for A and B are 60 miles apart, and similarly the broadcast locations for B and C are 60 miles apart—sufficiently close that their signals would create interference with each other. However, the broadcast locations for A and C are 120 miles apart—sufficiently distant that television stations A and C can coexist.

Similar examples can also arise completely outside the radiofrequency area. Suppose that a franchisor wishes to allocate franchises by auction. At the same time, franchisees may require some degree of geographic exclusivity for their franchises to be profitable (e.g., the franchisor may want to guarantee to each franchisee that no other franchise will be opened within a 60-mile radius). The feasibility constraint in the franchise auction would then operate in exactly the same way as in the television license auction.

In a real-world example, ICANN (the Internet Corporation for Assigned Names and Numbers) began accepting applications in 2012 for new generic top-level domains. Unique qualifying applications for different domain names would be granted, but ICANN's policy was to avoid granting applications for names that would be confusingly similar to one another. For example, the evaluators might decide that the domains .car and .cars are too close to each other and could not both be issued. They might also decide that the domains .cars and .cares are too close to each other and could not both be issued. However, the evaluators might feel that the domains .car and .cares are sufficiently far apart that they can coexist.

Particular implementations of competition policy in spectrum auctions can also produce non-transitive conflicts. Consider an auctioneer seeking to allocate two licenses (East and West) among three bidders: incumbent A, who is interested only in East; incumbent B, who is interested only in West; and entrant C. Suppose that the auctioneer utilizes a "floating" set-aside, in which one of the two licenses is reserved for an entrant, but it is not pre-specified which license is reserved. (Often a floating set-aside is equivalent to an aggregate cap on what the incumbents as a group can win, in this case, one license.) If entrant C is only interested in West, then the conflict depicted in Figure 1 emerges from this scenario: the common interest in the West license creates a conflict between B and C, while the floating set-aside creates a conflict between A and B, but there is no conflict between A and C.[28]

Observe that, in any scenario captured by Figure 1, the bidders for A and C are perfect complements. Indeed, the scenario creates exactly the same complementarities as in the standard local-local-global model (where the bidder for B, which conflicts with both A and C, takes the role of the "global" bidder). More complex graphs of conflict interactions will create more complex patterns of complementarities. Similar effects may also occur in other environments with complex feasibility constraints, such as those present in electricity grids or when allocating airport landing/takeoff slots. Following the same reasoning as in previous sections, such settings can cause anomalies for the VCG mechanism. Zero-revenue and other non-core outcomes may arise. Winner collusion and even loser collusion become possible. The results may look eerily similar to those of assignment stages, even though there is no contiguity requirement imposed.

IX. Conclusion

The possibility of a non-core, zero-revenue outcome of the VCG mechanism has been known for almost 20 years. In this article, we demonstrate that this anomaly is not merely a theoretical curiosity but a real problem, as we document its occurrence in the field for the first time: in the high-profile, high-stakes Broadcast Incentive Auction.

Knowing the theory that we have developed herein, it is not at all surprising that zero-revenue outcomes manifested themselves in the assignment stage of a spectrum auction. An assignment stage that respects contiguity is subject to two different types of complementarities. First, a bidder is deprived of a desired assignment if each of two opponents demands frequencies overlapping the desired assignment (a *direct* complementarity). Second, the bidder is also deprived of the desired assignment if one of the opponents demands frequencies which, while not overlapping, render it infeasible for the bidder to win the desired assignment due to contiguity (an *indirect* complementarity). When either of these complementarities come into play in the VCG mechanism, neither opponent is individually responsible for depriving the bidder of its choice—and so both escape bearing the opportunity cost of depriving the bidder of its choice.

Just because bidder complementarities (and non-core outcomes) are exacerbated by assignment stages in theory and just because non-core outcomes are observed there in practice does not imply that the two-stage approach should be abandoned. Quite the opposite, it has numerous practical advantages, including the mitigation of anti-competitive strategies and the simplification of bidding. For what it is worth, the spectrum regulators of the US, UK, Canada and Australia have adopted allocation stages for generic blocks, followed by assignment stages, in most of their recent auctions. But our findings, that

[28] At the same time, observe that certain other implementations of competition policy will never by themselves create bidder complementarities, for example, conventional "spectrum caps" (i.e., individual limits on the number of licenses that bidders can win) and fixed set-asides (reservations of particular licenses for entrants).

bidder complementarities both should be expected in assignment stages and do empirically occur there, highlight the practical limitations of the VCG mechanism. One needs to be cautious about utilizing VCG for what would otherwise seem to be a natural application, knowing that revenues may become uncompetitively low and that the mechanism may become susceptible to loser collusion and a laundry list of other vulnerabilities.

What about alternatives to the VCG mechanism, including the first-price menu auction (which is itself a core-selecting mechanism) and various minimum-revenue core-selecting auctions (e.g., the "nearest-VCG" mechanism)? Our results indicate the prevalence of bidder complementarities in assignment stages. From first principles, we know that a complementarity among costly actions of agents trying to achieve a common goal must give rise to a free-rider problem for *some* agents. In the VCG mechanism, the full burden of the free-rider problem is borne by the auctioneer, who accepts the uncompetitively low revenues of a non-core outcome. Switching to a core-selecting mechanism simply pushes the free-rider problem onto the complementary bidders, who will respond by bidding conservatively in an attempt to free-ride on one another, and the auctioneer would still face some degree of depressed revenues.[29] To summarize, the systematic presence of bidder complementarities leads generally to uncompetitively low revenues, irrespective of the payment rule.

But even if all auction formats suffer from low revenues, one should not prematurely conclude that the VCG mechanism is the right choice for an assignment stage. It does create the casual appearance of offering superior efficiency properties relative to core-selecting auctions (which are generally not incentive compatible), but only if the VCG mechanism is taken at face value. However, as we have seen, low revenue is only one of several disturbing pathologies of the VCG mechanism under bidder complementarities. The possibility of non-core outcomes also opens the door to winner collusion, loser collusion, and cost-free predation. Given these anomalies, any expectation that the VCG mechanism would elicit truthful values or achieve efficiency in the wider sense is probably naïve. Making matters worse, assignment stages in spectrum auctions with regional licenses are routinely run sequentially (one region after another), with the same bidders participating repeatedly and having opportunities to learn how to exploit these vulnerabilities.

Thus, it appears that there are advantages and disadvantages both to the VCG mechanism and to alternative auction formats for assigning frequencies. In practice, all assignment stages of which we are aware, prior to the Broadcast Incentive Auction, used a core-selecting mechanism (typically a weighted or unweighted "nearest-VCG" mechanism). The reader may wonder why the FCC broke with precedent and instead adopted a VCG mechanism for the assignment stage. The explanation is rather auction-specific. The amount of spectrum to be repurposed in the Broadcast Incentive Auction would be endogenously determined and, in order for a given quantity of spectrum to be cleared, the forward auction's revenues would be required to exceed the reverse auction's clearing costs. This balanced-budget constraint would be based solely on the first, allocation stage and would not include revenues from the second, assignment stage (which had not yet occurred). There were concerns that assignment stage payments might divert revenues away from the allocation stage, reducing the probability of meeting the balanced-budget constraint. As a result, the lowest coherent payment rule was adopted for the assignment stage.

[29] Ausubel and Baranov (2020).

We conclude this article with three observations: First, as of this writing, the FCC has conducted three more spectrum auctions taking the two-stage approach of an allocation stage for generic spectrum followed by an assignment stage for assigning specific frequencies—and it has proposed rules for a fourth such auction. Even though one of these auctions also has a revenue requirement with the potential to be binding, the FCC chose to utilize a core-selecting mechanism (rather than a VCG mechanism) for the assignment stage of all four of these auctions.

Second, while the contiguity preference with respect to frequency assignments on which we have focused is prevalent in spectrum auctions, similar value structures may arise naturally in other settings where other dimensions such as time may take on the same role as radiofrequency. For example, many generators in electricity auctions have preferences for contiguous dispatch times, in order to minimize ramp-up and ramp-down costs. Similarly, when government securities and other financial products transact in an auction, different buyers and sellers may wish to borrow or lend in different time periods, but most participants are likely to value contiguous time periods. As in this article, contiguity restrictions will give rise to bidder complementarities, creating similar anomalies and raising similar concerns.

Third, while the indirect complementarities encountered in this article were driven by a particular contiguity restriction, we have seen in Section VIII that similar phenomena can occur whenever a complex feasibility constraint is applied. As such, the clear message of this article that widespread complementarities may prevent the VCG mechanism from being a panacea can perhaps be taken as a more universal warning to mechanism designers.

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