Core-Selecting Auctions with Incomplete Information*

Lawrence M. Ausubel† and Oleg Baranov‡

February 17, 2018

Abstract

Core-selecting auctions were proposed as alternatives to the Vickrey-Clarke-Groves (VCG) mechanism for environments with complementarities. In this paper, we consider a simple incomplete-information model that allows correlations among bidders' values. We do a full equilibrium analysis of three core-selecting auction formats as applied to the "local-local-global" model. We show that seller revenues and efficiency from core-selecting auctions can improve as correlations among bidders' values increase, producing outcomes that are closer to the true core than are the VCG outcomes. Thus, there may be good reasons for policymakers to utilize core-selecting auctions rather than the VCG mechanism in realistic environments.

Keywords: Core-selecting auction, combinatorial auction, Vickrey auction, VCG mechanism, spectrum auction

---

*We gratefully acknowledge the support of National Science Foundation Grant SES-09-24773. We are also grateful to Peter Cramton, Paul Milgrom, John Rust, Alex Teytelboym and Dan Vincent for valuable conversations and to seminar participants at the University of Maryland, Carnegie Mellon University, the Econometric Society World Congress, the NBER Market Design Working Group Meeting, the University of Colorado, the New Economic School and the INFORMS Annual Meetings for helpful comments. All errors are our own.

†Department of Economics, University of Maryland, College Park, MD 20742, USA. Email: ausubel@econ.umd.edu

‡Department of Economics, University of Colorado, Boulder, CO 80309, USA. Email: oleg.baranov@colorado.edu

The celebrated Vickrey-Clarke-Groves (VCG) mechanism has the attractive property that truth-telling is a dominant strategy for all participants, leading to efficient allocations.[1] However, there is a longstanding theoretical critique warning us of several reasons to be wary of the Vickrey auction in environments with complementarities. First, the VCG mechanism may generate low revenues and "unfair" outcomes in the sense that there may be losing bidders willing to pay more than the winners' payments. Second, VCG outcomes may be non-monotonic in the sense that increasing the number of bidders may reduce the seller's revenues. Third, the VCG mechanism may be especially vulnerable to unusual forms of collusive behavior, including collusion by losing bidders and shill bidding. The general explanation for these shortcomings is that the presence of complementarities makes it possible for the VCG outcome to lie outside of the core.[2]

Bidding in recent spectrum auctions demonstrates that VCG outcomes outside of the core are not merely theoretical curiosities but are practical real-world concerns. In two prominent combinatorial clock auctions in which spectrum was licensed on a regional basis, the actual bids submitted during the allocation stage produced VCG outcomes that lay outside the core.[3] More strikingly, the 2016-17 FCC Incentive Auction used the VCG mechanism for its assignment phase, in which winning allocations of generic spectrum were assigned to physical frequencies. There were 228 repetitions of the VCG mechanism conducted for different regions of the US. Of these 228 instances, there were three occurrences of *zero-revenue* VCG outcomes and a total of 38 instances in which the VCG result lay outside the core.[4]

Theoretical concerns about the performance of the VCG mechanism in the presence of complementarities led a series of authors to propose remedies. Ausubel and Milgrom (2002) formulated a specific alternative procedure known as the ascending

[1] The VCG mechanism was developed in the work of Vickrey (1961), Clarke (1971) and Groves (1973). Throughout this paper, we will use the terms "VCG mechanism" and "Vickrey auction" interchangeably.

[2] The core is the subset of allocations in payoff space that are feasible and unblocked by any coalition. When the auction outcome is not in the core, there exists a coalition of bidders willing to renegotiate the outcome with the seller, leading to instability. See Ausubel and Milgrom (2002, 2006) for these critiques and for a general characterization.

[3] These two auctions were the 2014 Canadian 700 MHz Auction ($5.27 billion in revenues) and the 2015 Canadian 2500 MHz Auction ($755 million in revenues). Note that the rules of these auctions included use of a core-selecting mechanism, so the actual outcomes were in the core relative to these bids. Also note that the bids from these auctions were disclosed after the auction on the Canadian regulator's website, `https://www.ic.gc.ca/eic/site/smt-gst.nsf/eng/h_sf01714.html`, but to the authors' knowledge, these have been the *only* combinatorial clock auctions with package bidding and regional licenses to date in which the regulator disclosed the bids.

[4] See Section 8 of Ausubel, Aperjis and Baranov (2018). Most prior spectrum auctions that utilized assignment rounds had used core-selecting auctions to determine the prices of bidders' physical assignments. However, the FCC Incentive Auction used the VCG mechanism, in order to minimize the extent to which revenues would be diverted from the main stage of the auction to the assignment stage.

proxy auction. Day and Raghavan (2007) and Day and Milgrom (2008) proposed an entire class of alternative payment rules that have become known as core-selecting auctions. As in the VCG mechanism, bidders submit package bids in core-selecting auctions, and the auctioneer determines the combination of bids that maximizes the total value subject to feasibility. Unlike the VCG mechanism, the core-selecting pricing rule assures that the auction outcome is always in the core relative to the reported values. Despite their recent development, core-selecting auctions are already being actively used today in major auction applications, mainly as components of the combinatorial clock auction design for spectrum auctions.[5]

In complete-information settings, core-selecting auctions have been shown to have efficient equilibria that generate higher revenues than the VCG mechanism. However, the complete-information assumption is critical for these results, and much of the motivation for using the VCG or other auction mechanisms is that bidders possess incomplete and asymmetric information. With incomplete information, bidder incentives for truthful bidding become important. Day and Raghavan (2007) and Day and Milgrom (2008) proposed a class of bidder-optimal core-selecting auctions that is shown to minimize the maximal gains from deviations from truthful bidding.[6]

In our paper, we consider a stylized class of models with incomplete information which is colloquially known as the local-local-global (LLG) model.[7] In this model, the auctioneer wishes to allocate two items. Two "local" bidders are interested in a single item each and they jointly compete against the "global" bidder who values both items as perfect complements. With truthful bidding, the VCG outcome falls outside the core whenever local bidders win (their total payment is less than the value of the global bidder). At the same time, the use of a core-selecting auction leads to a severe free-rider (or "threshold") problem between local bidders who face a common core-constraint imposed by the global bidder.

It is reasonable to expect some value correlations among all bidders in this model. To simplify the analysis, we abstract away any global-local correlations; including them would merely introduce correlations between competing parties that are well understood in the literature. However, we preserve the local-local correlations, as they are essential for obtaining our key insights. These are value correlations between two local bidders who need to cooperate in order to outbid the global bidder,

[5]For examples of recent spectrum auctions that have used a core-selecting component, including the detailed auction rules and their results, see `https://www.ofcom.org.uk/spectrum/spectrum-management/spectrum-awards/awards-archive/800mhz-2.6ghz` and `http://www.ic.gc.ca/eic/site/smt-gst.nsf/eng/h_sf10598.html`. For discussions, see Cramton (2013) and Ausubel and Baranov (2017).

[6]"Bidder-optimal core-selecting auctions" are mechanisms that always choose allocations that minimize revenues within the set of core allocations and, hence, are optimal from the bidder's viewpoint.

[7]To the best of our knowledge, the model first appeared in Krishna and Rosental (1996).

and the structure of the auction pricing rule can affect their ability to cooperate. While in general, local-local correlations might be less likely than global-local correlations, there are a number of important practical settings (including the reverse part of the FCC Incentive Auction[8] and the assignment phase of any spectrum auction[9]) where we would expect local-local correlations to be prevalent.

Several papers study the equilibrium performance of various bidder-optimal core-selecting auctions using the same LLG setting. Erdil and Klemperer (2010) define a class of payment rules referred to as "reference rules" and argue that they reduce the marginal incentive to deviate as compared to other core-selecting payment rules. Sano (2010) provides an equilibrium analysis of a sealed-bid version of the "ascending" proxy auction and Sano (2012) studies the dynamic version of the same payment rule. Hafalir and Yektaş (2015) characterize an auction that minimizes the core deviation using the mechanism design approach.

We perform an equilibrium analysis for three bidder-optimal core-selecting rules: the proxy rule proposed by Ausubel and Milgrom (2002), the nearest-VCG rule due to Day and Cramton (2012), and a nearest-bid rule that we are introducing here for completeness. While the performance of these rules cannot be ranked in general, we show that an unambgiuous ranking exists under the assumption of perfect correlation. The proxy rule is shown to depend the least on the bidder's own bids, thus providing the best incentives and leading to the best performance. In contrast, the nearest-bid rule is shown to induce almost first-price-like incentives for bid shading, producing the worst performance.

An important negative result in the literature is due to Goeree and Lien (2016). For independent private value settings, the authors show that any core-selecting

[8]Consider a reverse auction in which the auctioneer needs to repack three TV stations into two adjacent channels in a geographic area (and buy out any stations that cannot be repacked). There is one full-power station, which creates both co-channel and adjacent-channel interference when broadcasting, so putting this station on either channel renders the other channel unusable. There are also two low-power stations, which create only co-channel interference, so they can coexist on adjacent channels. Together, the three stations interact exactly as in the LLG model. However, the local bidders have more in common with each other than with the global bidder; indeed, the local bidders are low-power stations that are essentially in a different industry than the full-power station. Consequently, it is plausible that there may be a high degree of correlation between the two local bidders' values, while very little correlation with the global bidder's value.

[9]Consider an assignment phase that allocates four contiguous spectrum licenses, A, B, C and D, among three bidders. One bidder (global) has the right to receive two contiguous licenses and has a strong preferences for winning AB (perhaps, the global bidder already owns licenses adjacent to A from below and winning AB would create a large contiguous segment). Two other bidders (locals) have the right to receive one license each and have preferences for preventing the global bidder from obtaining AB. This can be accomplished by a local bidder winning either A or B. Thus, the three bidders interact as in the LLG model, where the winning side receives AB and the losing side gets CD. Furthermore, it is quite plausible that the values of the local bidders (the benefit of depriving the global bidder of a large contiguous segment) are strongly correlated with each other while being relatively independent of the global bidder's value.

auction that has an efficient equilibrium generates the same expected revenue as the VCG mechanism (due to the revenue equivalence theorem). This result implies that "truly core-selecting auctions" (mechanisms that select core outcomes with respect to true values) do not generally exist. In addition, they provide an illustrative example using the LLG setting where the nearest-VCG rule produces both lower efficiency and lower revenues than the VCG mechanism.

Our findings generally go in the opposite direction of those of Goeree and Lien (2016). First, we show that the revenue comparison in their illustrative example relies on specific distributions and can be reversed. Second, we find that the presence of correlations can dramatically affect the equilibrium performance of core-selecting auctions, both in the positive and negative directions, depending on the structure of the pricing rule. Most remarkably, under certain assumptions, the proxy rule is shown to achieve the first-best performance by inducing truthful bidding in the unique equilibrium. Thus, our analysis shows that core-selecting auctions can perform reasonably well in nontrivial, empirically-relevant settings.

Basing our analysis on value correlations might initially come across as inapt. We know from Crémer and McLean (1985) and McAfee and Reny (1992) that all informational rents can be extracted from bidders with correlated types, enhancing both efficiency and revenues. However, such optimal mechanisms employ transfers to administer punishments that violate ex post rationality, and such optimal mechanisms can be sensitive to the exact information structure. We do not consider such mechanisms here. Instead, our analysis is more in the spirit of the so-called "Wilson doctrine," in that we analyze specific sets of auction rules from the literature that are similar to rules in actual use. "The rules of these markets are not changed daily as the environment changes; rather they persist as stable, viable institutions" (Wilson, 1987, p. 36). They are reasonably robust to small changes in the information structure and they are always ex post rational for bidders. Furthermore, observe that correlations bring our setting closer to the complete-information environment where core-selecting auctions perform well. The underlying reason for the good performance is the knowledge of the efficient outcome: knowing which side of the market, global or local, must win. However, allowing local-local correlations does not enhance this knowledge in our model; instead, it only affects the ability of local bidders to overcome their free-rider problem under specific rules of a particular auction.

Our paper proceeds as follows. Section 1 contains the model and describes various core-selecting auctions. Section 2 derives the intuitive form for the optimality conditions that enable the equilibrium analysis. The case of imperfect correlation is considered in Section 3, and the case of perfect correlation is considered in Section 4. Discussion of the results is provided in Section 5 and Section 6 concludes. Technical proofs can be found in the Appendix.

1 Model

Two items are offered for sale. There are two local bidders, 1 and 2, who are interested in only one item and receive no extra utility from acquiring the second item. Their values are denoted v_1 and v_2, respectively. There is also a global bidder who wants to acquire both items and obtains no utility from owning just one item. Her value for the pair of items is denoted u. Bidders are risk neutral and have quasilinear utilities: the payoff of local bidder i, if she wins one unit at price p_i, is $v_i - p_i$; and the payoff of the global bidder, if she wins both units at a total price of p, is $u - p$. The model can be interpreted as an environment where the items are taken either to be homogeneous or heterogeneous.[10]

The value u of the global bidder is independently drawn from the distribution on $[0, \bar{u}]$ defined by a cumulative distribution function $G(u)$ with atomless probability density function $g(u)$. For the local bidders, we consider two correlation models. In the first model, referred to as the *standard model*, values v_1 and v_2 of local bidders are given by a weighted sum of the common component s and a private component z_i, i.e., $v_i = \gamma s + (1-\gamma) z_i$ for both bidders. Parameter $\gamma \in [0,1]$ controls the strength of correlation between the local bidders' values. In the second model, referred to as the *analytical model*, parameter γ represents the probability that both local bidders have exactly the same value. Formally, with probability γ, the values of both local bidders are equal to the common component s, and with probability $(1-\gamma)$, each value v_i is equal to the corresponding private component z_i. In both models, each local bidder observes only its own realization of v_i. Variables s, z_1, z_2 are drawn independently from a distribution on $[0, \bar{v}]$ defined by a cumulative distribution function $F(v)$ with atomless density $f(v)$. Throughout the paper, we assume that $\bar{u} \geq 2\,\bar{v}$.

The main advantage of the analytical model is that it avoids dealing with convolutions of random variables while closely following the conditional distributional properties of the standard model. Conditional CDFs $F(v_j|v_i = x)$ in both models are the same when values are fully independent ($\gamma = 0$) and perfectly correlated ($\gamma = 1$). Conditional CDFs of local bidder i who observed $v_i = 0.5$ for several non-trivial values of correlation parameter $\gamma \in (0,1)$ are shown in Figure 1 (assuming that F is uniform).

It is important to note that in the analytical model, v_i and v_j are not affiliated random variables as long as $\gamma > 0$.[11] Nevertheless, for any $y \geq x$, $F(v_j|v_i = y)$ first-

[10]In the former case, local bidder i derives positive utility v_i from winning either item, and the global bidder exhibits classic increasing returns to scale. In the latter case, there are two heterogeneous items, A and B; local bidder 1 obtains positive utility only from A, local bidder 2 obtains positive utility only from B and the global bidder views A and B as perfect complements.

[11]Consider $x > y > z$ and let $\Psi(.,.)$ denote the joint probability of v_i and v_j. Then $(y,y) \vee (x,z) = (x,y)$ and $(y,y) \wedge (x,z) = (y,z)$, but $\Psi(x,y) \cdot \Psi(y,z) < \Psi(y,y) \cdot \Psi(x,z)$, contradicting the affiliation inequality.

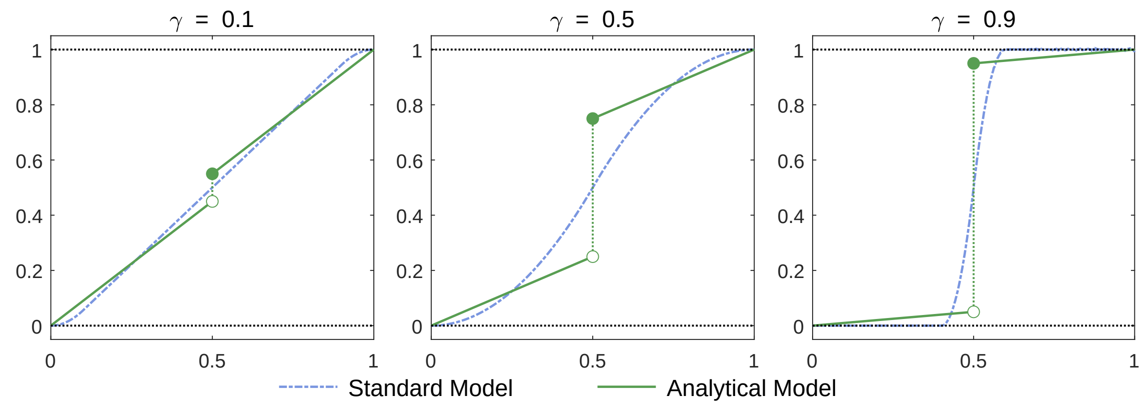

Figure 1: $F(\,v_j\,|\,v_i\;=\;0.5\,)$ *for* $\gamma\in\{0.1,\,0.5,\,0.9\,\}$ *(drawn for* $F(x)\;=\;x$*)*

order stochastically dominates $F(v_j|v_i = x)$, so v_i and v_j are positively depend on each other. The failure of affiliation would prevent some of the results in the theory of single-item auctions from going through. For example, the analytical model creates a probability mass point in the conditional distribution function which would result in nonexistence of the equilibrium in a standard first-price auction for a single item.[12] This concern is not an issue in the current framework due to a different structure of competition. Local bidder 1 is not competing with local bidder 2; rather they jointly compete against the global bidder.

We consider a restricted class of sealed-bid auctions in which each bidder is allowed to place only one bid. In particular, each local bidder i bids $b_i \geq 0$ for the item she values, and the global bidder bids $B \geq 0$ for the package.[13]

In each of the core-selecting auctions considered, the auctioneer selects a value-maximizing allocation. In the LLG model, only two outcomes are possible: the global bidder wins both items when $B > b_1 + b_2$, and the local bidders win one item each when $B < b_1 + b_2$. Ties are resolved using a fair randomizing device. The payment of each winner depends on the pricing rule. We denote $p(b_1, b_2, B)$ a payment vector associated with bids b_1, b_2 and B for a particular pricing rule.

All auction mechanisms analyzed in this paper, other than the VCG mechanism, satisfy the following definition. A *core-selecting auction* is a mapping from bids to allocations and payments such that the payoffs resulting from every bid profile are elements of the core. Core-selecting auctions that always choose a bidder-optimal element of the core are referred to as *bidder-optimal core-selecting auctions*.

[12]Consider a symmetric first-price auction with two bidders whose values are correlated in the same way. If one bidder knows that the other bidder has the same value with a positive probability, her best response fails to exist.

[13]See Ott and Beck (2013) for the analysis of the LLG model when bidders are allowed to submit bids on bundles that include unwanted items.

In the LLG setting, the payment rule of the VCG mechanism is given by:

$$p(b_1, b_2, B) \;=\; \begin{cases} (p_1^V, \, p_2^V, \, 0) & \text{if} \quad B \;<\; b_1 \;+\; b_2 \\ (0, \, 0, \, b_1 \;+\; b_2) & \text{if} \quad B \;>\; b_1 \;+\; b_2 \end{cases} \tag{1.1}$$

where $p_1^V \;=\; \max\{0, B - b_2\}$ and $p_2^V \;=\; \max\{0, B - b_1\}$. A pricing rule associated with a core-selecting auction is given by:

$$p(b_1, b_2, B) \;=\; \begin{cases} (p_1, \, p_2, \, 0) & \text{if} \quad B \;<\; b_1 \;+\; b_2 \\ (0, \, 0, \, P) & \text{if} \quad B \;>\; b_1 \;+\; b_2 \end{cases} \tag{1.2}$$

such that $p_1 \in [p_1^V, b_1]$, $p_2 \in [p_2^V, b_2]$, $p_1 + p_2 \;\geq\; B$ and $P \in [b_1 + b_2, B]$. Finally, $p_1 + p_2 \;=\; B$ and $P \;=\; b_1 + b_2$ in any bidder-optimal core-selecting auction. In the paper, we explicitly consider several bidder-optimal core-selecting auctions. In all of them, the global bidder pays $b_1 + b_2$ upon winning. When local bidders win, they split the total payment of B as follows (it is assumed that $b_1 \;\geq\; b_2$ for convenience):

(1) **Proxy Rule** *(Nearest-Zero Rule)*

The "ascending" proxy auction was suggested by Ausubel and Milgrom (2002). For the LLG model, it is equivalent to selecting a point in the bidder-optimal core which is the closest to zero and is summarized as follows:

$$p(b_1, b_2, B) \;=\; \begin{cases} \left(\frac{1}{2}B, \, \frac{1}{2}B, \, 0\right) & \text{if} \quad 0 \;\leq\; B \;\leq\; 2\,b_2 \\ (B \;-\; b_2, \, b_2, \, 0) & \text{if} \quad 2\,b_2 \;<\; B < \; b_1 \;+\; b_2 \end{cases} \tag{1.3}$$

(2) **Nearest-VCG Rule**

The nearest-VCG pricing rule was introduced by Day and Cramton (2012), superseding the suggestion of minimizing the maximum deviation from the VCG payments that was made in Day and Raghavan (2007). The central idea of this rule is to select the bidder-optimal core allocation that minimizes the Euclidean distance from the VCG outcome. For the LLG model, this rule is summarized as follows:

$$p(b_1, \, b_2, \, B) \;=\; (p_1^V \;+\; \Delta, \, p_2^V \;+\; \Delta, \, 0) \tag{1.4}$$

where $\Delta \;=\; \frac{1}{2}\,(B \;-\; p_1^V \;-\; p_2^V)$.

(3) **Nearest-Bid Rule**

The nearest-bid description corresponds to the bidder-optimal core allocation that is the closest to the winners' bids.[14] For the LLG model, the rule is

[14]The nearest-bid rule can be articulated as follows. In case of winning, each local bidder pays her bid minus a discount. The amount of the discount is half of the "money left on the table", i.e. $\frac{1}{2}[b_1 + b_2 - B]$. When bids are too different, the amount of the discount can exceed the bid amount in which case the bidder pays zero.

summarized as follows:

$$p(b_1, b_2, B) = \begin{cases} (B, 0, 0) & \text{if} \quad 0 \leq B \leq b_1 - b_2 \\ (b_1 - \Delta, b_2 - \Delta, 0) & \text{if} \quad b_1 - b_2 < B < b_1 + b_2 \end{cases} \tag{1.5}$$

where $\Delta = \frac{1}{2}(b_1 + b_2 - B)$.

2 Preliminary Analysis

We start by introducing the *pivotal pricing property*. Suppose that the set of value-maximizing allocations includes several allocations and there is a bidder who can either win a non-empty set of items or win nothing depending on a realization of a tie-breaking rule. If the auction satisfies the pivotal pricing property, then such bidder must necessarily pay its bid amount in case she is awarded a non-empty set of items.

It is easy to verify that the VCG mechanism satisfies this property. It can be shown that this pivotal pricing property holds for all core-selecting auctions in the general setting. Here we state it for the LLG setting (where the tie-breaking occurs only when $b_1 + b_2 = B$ and this property is trivially satisfied).

Lemma 1. *Every core-selecting auction satisfies the pivotal pricing property in the LLG model.*

We further restrict the class of bidder-optimal core-selecting auctions that we consider by imposing the following *regularity conditions*. For local bidder i, and any bid vector (b_1, b_2, B) such that local bidders win, her price function $p_i(b_1, b_2, B)$ is continuous in all bids and differentiable in her own bid. In addition, for any bid vector, her marginal payment is nonnegative (i.e., a bid increase cannot decrease the payment). It is easy to verify that the VCG mechanism and core-selecting auctions (1.3) – (1.5) satisfy these regularity conditions.

It is well-known that truthful bidding is a weakly dominant strategy for all bidders in the VCG mechanism. Lemma 2 identifies weakly dominated strategies in bidder-optimal core-selecting auctions in the LLG model.

Lemma 2. *Suppose that a bidder-optimal core-selecting auction satisfies the regularity conditions. Then, for the global bidder, bidding her value is a weakly dominant strategy; and for a local bidder, bidding above her value is a weakly dominated strategy.*

For the equilibrium analysis, we assume that the global bidder always bids according to her weakly dominant strategy $B(u) = u$ and each local bidder i bids according to $\beta_i(v_i)$ such that $\beta_i(v_i) \leq v_i$ for all $v_i \in [0, \bar{v}]$. Denote $\Phi_i(b_i, v_i)$ and

$\phi_i(b_i, v_i)$ the probability of winning and its marginal for local bidder i when she bids $b_i \in [0, v_i]$:

$$
\begin{aligned}
\Phi_i(b_i, v_i) &= Pr(b_i + \beta_j(v_j) \geq u) \\
\phi_i(b_i, v_i) &= \frac{\partial\, \Phi_i(b_i, v_i)}{\partial\, b_i}
\end{aligned}
\tag{2.1}
$$

In addition, denote $P_i(b_i, v_i)$ and $MP_i(b_i, v_i)$ the expected payment and the expected marginal payment for local bidder i when she bids $b_i \in [0, v_i]$:

$$
\begin{aligned}
P_i(b_i, v_i) &= E\left[p_i(b_i, \beta_j(v_j), u) \,|\, v_i \right] \\
MP_i(b_i, v_i) &= E\left[\frac{\partial\, p_i(b_i, \beta_j(v_j), u)}{\partial\, b_i} \,\middle|\, v_i \right]
\end{aligned}
\tag{2.2}
$$

Next proposition simplifies the first-order optimality conditions for local bidders in all bidder-optimal core-selecting auctions that satisfy regularity conditions. Note that such auctions satisfy the pivotal pricing property by Lemma 1.

Proposition 1. *Suppose that a bidder-optimal core-selecting auction satisfies the regularity conditions. Then the optimality condition for choosing bid* $b_i \geq 0$ *for a local bidder* i *is given by:*

$$
\begin{gathered}
(v_i - b_i)\, \phi_i(b_i, v_i) \leq MP_i(b_i, v_i) \\
\textit{with equality when } b_i > 0
\end{gathered}
\tag{2.3}
$$

Intuitively, a small increase in b_i increases bidder's profit by allowing her to win in the pivotal state (while paying b_i due to the pivotal pricing property) at the cost of increasing her expected marginal payment in non-pivotal states. At the optimum, both effects have to be equal.

3 Imperfect Correlation ($\gamma < 1$)

This section contains equilibrium analysis for the case of imperfect correlation between values of two local bidders. We consider two ways of modeling this correlation. In the standard model, value of local bidder i is given by $v_i = \gamma\, s + (1-\gamma)\, z_i$ where z_i is a private component and s is a common component. In the analytical model, $v_1 = v_2 = s$ with probability γ and $v_1 = z_1$, $v_2 = z_2$ with probability $1-\gamma$. When $\gamma \in (0,1)$, the inference of bidder i about v_j given her own v_i differ between two models.

We consider each model in turn. For the analytical model, we find the symmetric equilibrium for each pricing rule assuming that the global bidder draws its value from a uniform distribution. For the standard model, we numerically approximate optimality conditions under the same assumptions and demonstrate that the resulting bidding functions are qualitatively very similar to the corresponding equilibria of the analytical model.

3.1 Analytical Model

In this section, we present our first main result (Theorem 1) that proves existence of the symmetric equilibrium for each pricing rule (including implicit formulas for equilibrium bidding functions) assuming that the global bidder draws its value from a uniform distribution. We present an example of the environment where equilibrium bidding functions can be obtained as closed-form solutions (Corollary 1). Then the implicit solutions from Theorem 1 are used to prove comparative static results (Corollaries 2 and 3).

Theorem 1. *Consider the analytical model with $G(u) = u/\bar{u}$. For each pricing rule (1) – (3), the unique symmetric Bayesian-Nash equilibrium exists and the equilibrium bidding function for local bidders is implicitly given by:*

(a) for the proxy rule

$$\beta(v) \;=\; \max\left\{0,\; \tilde{\beta}(v)\right\}, \tag{3.1}$$

where $\tilde{\beta}(v)$ solves

$$\tilde{\beta}'(v) \;=\; \frac{1}{\gamma \;+\; (1 \;-\; \gamma)\, F(v)} \qquad \text{and} \qquad \tilde{\beta}(\bar{v}) \;=\; \bar{v}\,; \tag{3.2}$$

(b) for the nearest-VCG rule

$$\beta(v) \;=\; \max\left\{0,\; \frac{2}{2 \;+\; \gamma}\left(v \;-\; \hat{v}\right)\right\}, \tag{3.3}$$

where $\hat{v} \in (0, \bar{v})$ is a unique solution to

$$\frac{\hat{v}}{(1 \;-\; F(\hat{v}))} \;=\; \frac{1 \;-\; \gamma}{2 \;+\; \gamma}\, E\Big(v \;-\; \hat{v}\,|\, v \;\geq\; \hat{v}\Big)\,; \tag{3.4}$$

(c) for the nearest-bid rule

$$\beta'(v) \;=\; \frac{1}{2 \;-\; (1 \;-\; \gamma)\, F(v)} \qquad \text{and} \qquad \beta(0) \;=\; 0. \tag{3.5}$$

The implicit solutions from Theorem 1 have closed forms for some distribution functions. For example, equilibrium bidding functions can be found analytically when $F(.)$ is a uniform distribution on $[0, 1]$ and $\bar{u} \geq 2$.

Corollary 1. *Consider the analytical model with $F(v) = v$ and $G(u) = u/\bar{u}$. The equilibrium bid function of local bidders in the symmetric Bayesian-Nash equilibrium is given by:*

(a) *for the proxy rule*

$$\beta(v) \;=\; \max\left\{0,\ 1 + \frac{ln(\gamma \;+\; (1 - \gamma)\, v)}{1 \;-\; \gamma}\right\}\;; \qquad (3.6)$$

(b) *for the nearest-VCG rule*

$$\beta(v) \;=\; \max\left\{0,\ \frac{2}{2 + \gamma}\left(v - \hat{v}\right)\right\} \qquad where \qquad \hat{v} \;=\; \frac{3 - \sqrt{9 - (1-\gamma)^2}}{1-\gamma}\;; \quad (3.7)$$

(c) *for the nearest-bid rule*

$$\beta(v) \;=\; \frac{1}{1 \;-\; \gamma}\left[\, ln(2) \;-\; ln(2 \;-\; (1 \;-\; \gamma)\, v)\right]. \qquad (3.8)$$

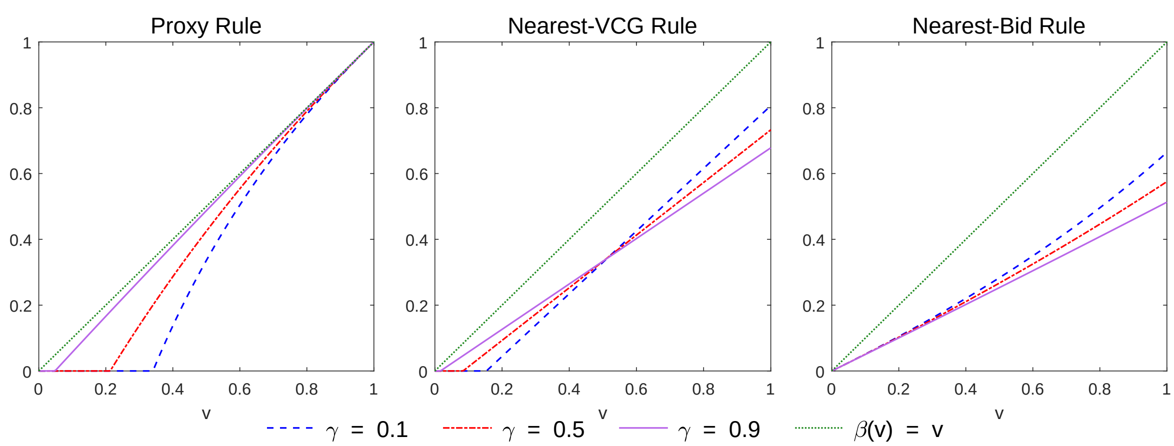

Figure 2: Equilibrium Bids of the local bidders (analytical model)

Equilibrium bidding functions from Corollary 1 are plotted in Figure 2 for $\gamma \in \{0.1,\, 0.5,\, 0.9\}$. First of all, note that different core-selecting rules result in a very different incentives depending on bidder's type. For example, the low (high) type bidder shades the most (least) under the proxy rule and the least (most) under the nearest-bid rule. Intuitively, the proxy rule creates first-price incentives for the low-type bidder and the second-price incentives for the high-type bidder. In contrast, the nearest-bid rule essentially subsidizes the low-type bidder at the expense of the high-type bidder, resulting in higher incentives to shade for the high-type.

As can be seen from Figure 2, an increase in correlation between values of local bidders has a dramatic effect on the equilibrium bidding functions. Under the proxy rule, bidders bid more competitively and, under the nearest-bid rule, bidders bid less competitively. Meanwhile, the effect on the nearest-VCG rule is ambiguous as low types bid more competitively and high types bid less competitively. Corollary 2 shows that this observed effect of correlation on the pricing rules is general. For two correlation levels, γ and γ', denote $\beta_\gamma(v)$ and $\beta_{\gamma'}(v)$ the corresponding equilibrium bidding functions for a particular pricing rule.

Corollary 2. *Consider the analytical model with $G(u) = u/\bar{u}$. If $\gamma \leq \gamma'$, then:*

(a) for the proxy rule

$$\beta_\gamma(v) \leq \beta_{\gamma'}(v) \qquad \forall\, v \in [0, \bar{v}]; \tag{3.9}$$

(b) for the nearest-VCG rule, there exists $\tilde{v} \in (\hat{v}(\gamma), \bar{v})$ such that

$$\begin{aligned} \beta_\gamma(v) &\leq \beta_{\gamma'}(v) \qquad \forall\, v \in [0, \tilde{v}], \\ \beta_\gamma(v) &\geq \beta_{\gamma'}(v) \qquad \forall\, v \in [\tilde{v}, \bar{v}], \end{aligned} \tag{3.10}$$

where $\hat{v}(\gamma)$ is the solution to (3.4);

(c) for the nearest-bid rule

$$\beta_\gamma(v) \geq \beta_{\gamma'}(v) \qquad \forall\, v \in [0, \bar{v}]. \tag{3.11}$$

Furthermore, we can rank equilibrium bids whenever underlying distributions can be ranked in the first-order stochastic dominance sense. Intuitively, when the underlying value distribution of local bidders is weaker, local bidders would bid more competitively in the equilibrium to account for the reduced free-riding opportunities. For two cumulative distribution functions defined on $[0, \bar{v}]$, F_1 and F_2, denote $\beta_{F_1}(v)$ and $\beta_{F_2}(v)$ the corresponding symmetric equilibrium bidding function for local bidders for each pricing rule.

Corollary 3. *Consider the analytical model with $G(u) = u/\bar{u}$. If F_1 first-order stochastically dominates F_2, then for each pricing rule (1) – (3),*

$$\beta_{F_1}(v) \leq \beta_{F_2}(v) \qquad \forall\, v \in [0, \bar{v}]. \tag{3.12}$$

3.2 Standard Model

In this section, we provide a short analysis of the standard model. In comparison with the analytical model, the standard model presents a major analytical challenge of dealing with a convolution of random variables since bidder i's inference about the common component s has to be inferred from her $v_i = \gamma\, s + (1-\gamma)\, z_i$ without observing z_i, and then used to infer $v_j|v_i = \gamma\, s|v_i + (1-\gamma)\, z_j$. Given this challenge, we employ numerical methods to approximate equilibria and then compare results to the ones obtained for the analytical model.

Using numerical methods, we solve for the bidding function that approximate optimality conditions (2.3) under the same assumptions that were used in Corollary

1 and Figure 2 (i.e., $F(v) = v$, $G(u) = u/2$ and $\gamma \in \{0, 1, 0.5, 0.9\}$). Results are plotted in Figure 3. Note that given the difference in conditional distributions, the bidding functions in Figure 3 may differ from the equilibrium bidding functions obtained for the analytical model when $\gamma \in (0, 1)$.[15]

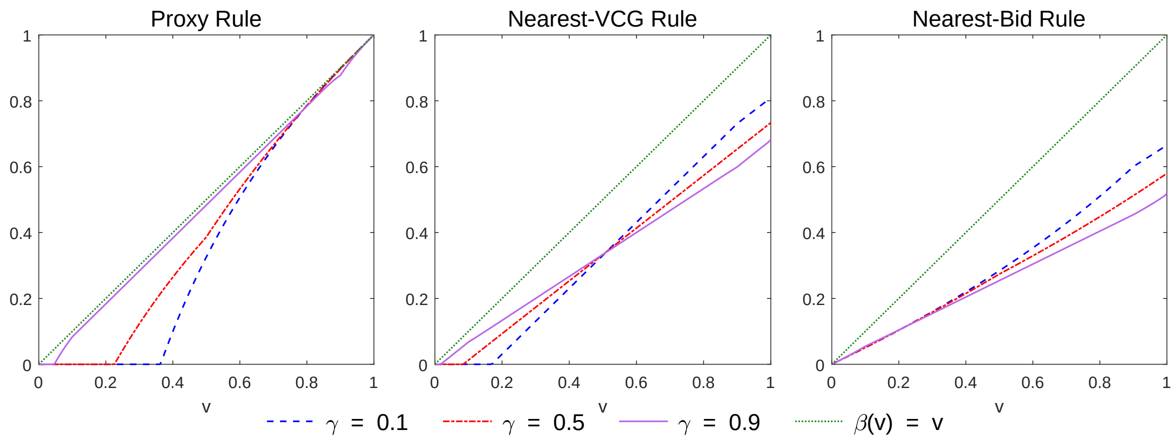

Figure 3: Equilibrium Bids of the local bidders (standard model)

Visual comparison confirms that solutions for the standard and analytical models are qualitatively identical with one noticeable difference — kinks in bid functions for the standard model at low and high values. The kinks are produced by the "inference corners" that are present in the standard model, but not in the analytical model.[16] We conclude that the analytical model fully captures all elements of the standard model while simultaneously providing tractable and well-behaved equilibria.

When values of local bidders are perfectly correlated ($\gamma = 1$), there is no difference between the standard and analytical models. We analyze the case of perfect correlation in the next section.

4 Perfect Correlation ($\gamma = 1$)

This section contains equilibrium analysis for the case of perfect correlation between values of two local bidders. When correlation is perfect, the standard and analytical models are identical. Also note that distribution $F(.)$ from which the common component is drawn has no effect on equilibria in this case.

[15]The quality of our numerical solutions is validated using true solutions for $\gamma = 0$. The average absolute difference between two solutions is $4.7e - 06$ for the proxy rule, $7.9e - 08$ for the nearest-VCG Rule, and $1.1e - 0.6$ for the nearest-bid rule in this case.

[16]For example, if $v_i = 0$, then $s = 0$ and the range for $v_j|v_i$ is $[0, (1-\gamma)\bar{v}]$. In contrast, the range for $v_j|v_i$ in the analytical model is always $[0, \bar{v}]$ as long as $\gamma < 1$. The presence of corners suggests major difficulties in obtaining analytical equilibria in the standard model.

Our first result shows that the symmetric equilibrium for the proxy rule (when it exists) involves truthful bidding by all bidders. The existence of such equilibrium for some environments is proved in Theorem 2 at the end of this section.

Proposition 2. *Consider the model with perfect correlation. For the proxy rule, the unique symmetric Bayesian-Nash equilibrium in pure strategies (when it exists) is given by:*

$$\beta^{Proxy}(v) \;=\; v \qquad \forall v \,\in\, [0, \bar{v}] \tag{4.1}$$

When the truthful equilibrium exists, the proxy rule delivers first-best performance by generating fair revenues while simultaneously achieving full efficiency. Let R denote expected seller revenue and Ef denote expected efficiency for a particular auction.

Corollary 4. *Consider the model with perfect correlation. If the symmetric equilibrium exists for the proxy rule, then the expected efficiency and seller revenue in the symmetric equilibrium are compared as follows:*

(a) for the truthful equilibrium of the VCG mechanism

$$Ef^{Proxy} \;=\; Ef^{VCG} \qquad \textit{and} \qquad R^{Proxy} \;>\; R^{VCG}; \tag{4.2}$$

(b) for any equilibrium of any bidder-optimal core-selecting auction that satisfies regularity conditions and achieves Ef and R

$$Ef^{Proxy} \;\geq\; Ef \qquad \textit{and} \qquad R^{Proxy} \;\geq\; R. \tag{4.3}$$

Next we show that the symmetric equilibrium bids for pricing rules (1) – (3) can be ranked.

Proposition 3. *Consider the model with perfect correlation and suppose that the symmetric equilibrium exists for pricing rules (1) – (3). Then for any $v \in (0, \bar{v}]$:*

$$\beta^{Proxy}(v) \;>\; \beta^{N-VCG}(v) \;>\; \beta^{N-BID}(v). \tag{4.4}$$

When these symmetric equilibria exist and unique, the bid ranking (4.4) implies that the proxy rule is strictly better than the nearest-VCG rule, which in turn is strictly better than the nearest-bid rule, in terms of both efficiency and seller revenues.

Now we present our second main result. Theorem 2 proves that for a particular class of distributions, auctions with pricing rules (1) – (3) each has a unique Bayesian-Nash equilibrium in pure strategies.

Theorem 2. *Consider the model with perfect correlation and $G(u) = (u/\bar{u})^\sigma$ where $\sigma > 1$. For each pricing rule (1) – (3), the unique Bayesian-Nash equilibrium exists and the equilibrium bid function (symmetric) for local bidders is given by:*

(a) for the proxy rule

$$\beta(v) \;=\; v\,; \tag{4.5}$$

(b) for the nearest-VCG rule

$$\beta(v) \;=\; \frac{\sigma}{1 \,+\, \sigma \,-\, 2^{-\sigma}}\, v\,; \tag{4.6}$$

(c) for the nearest-bid rule

$$\beta(v) \;=\; \frac{\sigma}{1 \,+\, \sigma}\, v\,. \tag{4.7}$$

Equilibria identified in Theorem 2 highlight the key distinction among considered core-selecting rules. Parameter σ controls the distributional strength of the global bidder in the relevant value range $[0, 2\,\bar{v}]$. Consider the standard single-item auction with two bidders. If value distribution of bidder 1 is made weaker, the equilibrium bid of bidder 2 shifts downward in the first-price auction and does not change in the second-price auction. The same effect is observed here. Similar to a second-price auction, the change in global bidder's strength (change in σ) has no effect on equilibrium bids of local bidders for the proxy rule. In contrast, σ affects equilibrium bids for two other rules, but the effect is weaker for the nearest-VCG rule.[17]

Theorem 2 is limited to environments where $G(u) = (u/\bar{u})^{\sigma}$ and $\sigma > 1$. The comparison among various core-selecting rules and the VCG mechanism is less certain when $\sigma \in (0, 1)$ due to existence of multiple equilibria. For example, truthful bidding is no longer an equilibrium for the proxy rule. Instead, there are two asymmetric pure-strategy equilibria where one local bidder bids truthfully and the other local bidder bids zero.[18] For the nearest-VCG rule, the symmetric equilibrium (4.6) exists for any $\sigma > 0$, but fully asymmetric and partially asymmetric equilibria are also possible. Finally, a pure-strategy equilibrium does not exist for the nearest-bid rule. As a result, it is possible that the proxy rule performs worse than the VCG or nearest-VCG rules under such assumptions.

There is another critique that can be made of the proxy rule.[19] The existence of the truthful equilibrium is very sensitive to the symmetry assumption. To demonstrate this, suppose instead that the values of local bidders were perfectly correlated but slightly asymmetric, in that $v_1 = v$ and $v_2 = (1-\epsilon)\,v$, where $v \in [0, \bar{v}]$ and $\epsilon > 0$. In order to maintain the existence of a truthful equilibrium in this asymmetric setting, the proxy rule would need to be modified such that local bidders split the total payment in the proportions $\left(\frac{1}{2-\epsilon}\,B, \frac{1-\epsilon}{2-\epsilon}\,B\right)$ when they win. If a symmetric

[17] To see this, consider $\sigma \to 0$. Then $\sigma/(1+\sigma) \to 0$ while $\sigma/(1+\sigma-2^{-\sigma}) \to 1/(1+ln(2)) \approx 0.59$. For the nearest-bid rule, the first-price effect dominates. In contrast, the first-price effect has limited impact on the nearest-VCG rule and dissipates for low σ.

[18] For the same setting, Sano (2012) reports the same fully asymmetric equilibria for a dynamic version of the proxy rule.

[19] We are grateful to an anonymous referee for providing this comment.

$\left(\frac{1}{2}B, \frac{1}{2}B\right)$ split were used, bidder 2 would have the incentive to bid less than her value, since her bid would bind when $(2 - 2\epsilon)\,v \leq u \leq (2 - \epsilon)\,v$.

5 Discussion

To illustrate the welfare properties of different pricing rules, we calculate expected seller revenue, efficiency and distance to the true core using symmetric equilibria identified in Theorems 1 and 2. Figure 4 plots results for each pricing rule assuming uniform distributions $F(v) = v$ and $G(u) = u/2$. The expected seller revenue is plotted as a percentage of the "fair" revenue ($\min\{v_1 + v_2, u\}$). Both the expected efficiency and expected distance to the true core are plotted as a percentage of the maximum value ($\max\{v_1 + v_2, u\}$).

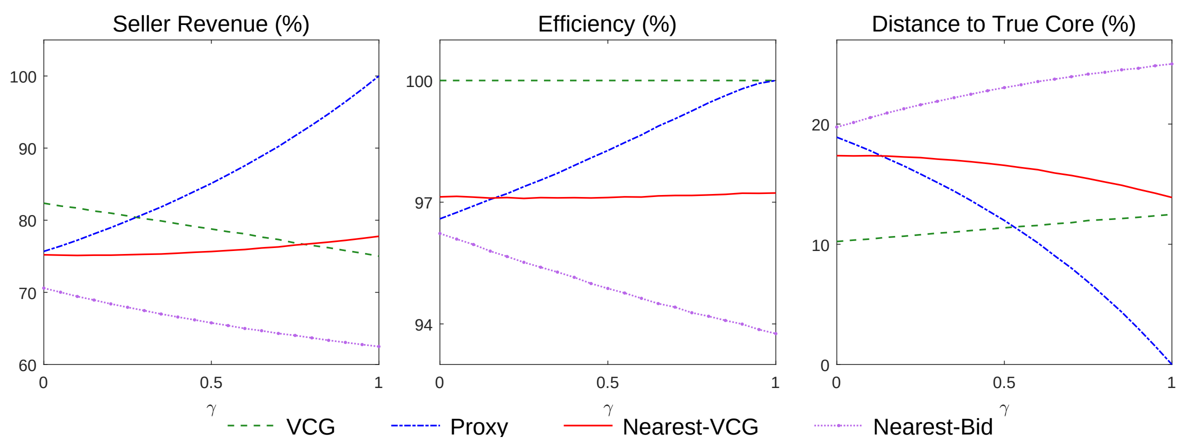

Figure 4: Revenue, Efficiency and Distance to True Core for $F(v) = v$ and $G(u) = u/2$

While all core-selecting auctions are generally inefficient, they can generate revenues that are more competitive than the VCG revenue. And significantly higher revenues can lead to a better overall performance as measured by the distance to the true core. Thus, in some environments there is a clear trade-off between using the VCG mechanism and core-selecting auctions.

Several general remarks are warranted here. First, the differential effects of correlation from Corollary 2 are clearly visible in Figure 4. The performance of the proxy rule improves rapidly as the correlation increases due to more competitive bidding, while the performance of the nearest-bid rule falls. Strong correlation also hurts VCG revenues, as it puts greater emphasis on low-revenue outcomes.

Second, no general revenue rankings are available between the VCG mechanism and the three core-selecting auctions. While the VCG rule is always efficient, any of the considered core-selecting rules can generate higher revenues under certain assumptions and, as a result, can also produce outcomes that are closer to the true core. The lack of general rankings is true even when local bidders' values are fully independent, the setting considered by Goeree and Lien (2016).[20]

Under perfect correlation, the considered core-selecting auctions can be compared when they happen to have unique equilibria. In such scenarios, the equilibrium bids and auction performance can be strictly ranked according to Proposition 3, based upon how much the pricing rule depends on bidders' own bids. Most remarkably, the unique equilibrium of the core-selecting auction with the proxy rule for certain distributions is associated with truthful bidding by all bidders. With truthful bidding being optimal, the auction generates fair revenues without sacrificing efficiency, and thus provides an example of a "truly" core-selecting auction.

Finally, while the proxy rule appears to dominate the nearest-VCG rule, it is also more sensitive to the assumptions. For example, for some parameter values, it supports fully asymmetric equilibria where one of the local bidders bids zero.

6 Conclusion

The previous literature has shown the VCG mechanism to have a variety of shortcomings in environments with complementarities, including the possibility of low or even zero revenues, non-monotonicity of revenues with respect to bids and number of bidders, and vulnerability to unusual forms of collusion such as shill bidding and collusion by losing bidders. These drawbacks may help to explain why this auction format — despite its attractive dominant-strategy property — is seldom used in practice. By contrast, the nearest-VCG pricing rule has been used numerous times in recent years for high-stakes spectrum auctions.

This paper studies a stylized LLG model in an environment with private information. Our analysis shows that in some environments, equilibrium outcomes of core-selecting auctions can be significantly closer to the true core than the VCG outcome. In particular, the comparison of some core-selecting auctions to the VCG mechanism dramatically improves in the presence of positive value correlations. Furthermore, we note that in important applications such as spectrum auctions, such positive correlations are likely to be present. Thus, unlike Goeree and Lien (2016), we conclude that there may be good reasons for policymakers to use core-selecting auctions rather than the VCG mechanism in their applications.

One of the novel aspects of our analysis is an "analytical model" of positive correlations. Our treatment admits simple closed-form solutions, without sacrificing any of the qualitative properties of the standard approach. Another potential application of our modeling is in economic experiments. The usual notion of value correlations expressed with equations may be overly complex for experiments,

[20] For example, outcomes under both the proxy rule and the nearest-VCG rule are closer to the true core than the ones under the VCG rule for all $\gamma \in [0,1]$ when $F(v) = v^3$ and $G(u) = u/2$.

perhaps resulting in subjects not understanding the scenario. By contrast, the verbal description of our analytical model — two values are the same with a given probability and are drawn independently otherwise — may be easier for subjects to understand and may thereby generate more consistent experimental results.

A Proofs

PROOF OF LEMMA 2. This follows a standard argument for the global bidder, who always pays $b_1 + b_2$ when she wins. For a local bidder, her payment is nondecreasing in her own bid due to regularity conditions. Then overbidding her value can only harm the local bidder. □

PROOF OF PROPOSITION 1. The optimality condition is given by: $\frac{\partial \pi_i(b_i,v_i)}{\partial b_i} = v_i\,\phi_i(b_i,v_i) - \frac{\partial P_i(b_i,v_i)}{\partial b_i} \leq 0$ (with equality when $b_i > 0$). Then equation (2.3) follows since $\phi_i(b_i,v_i) = \int_{v_j} f(v_j|v_i) g(b_i + \beta_j(v_j)) dv_j$ and $\frac{\partial P_i(b_i,v_i)}{\partial b_i} = MP_i(b_i,v_i) + b_i\,\phi_i(b_i,v_i)$ due to regularity conditions and the pivotal pricing property which ensures that $p_i(b_i, \beta_j(v_j), b_i + \beta_j(v_j)) = b_i$. □

PROOF OF THEOREM 1. By Proposition 1, the first-order conditions are given by (2.3). For all pricing rules, $\phi_i(b_i,v_i) = 1/\bar{u}$. Local bidder j follows strategy $\beta_j(v_j) \leq v_j$ which is strictly increasing on interval $[\hat{v}, \bar{v}]$ and is equal to zero on $[0, \hat{v}]$ where $\hat{v} \geq 0$. **(a):** For the proxy rule,

$$\frac{MP_i(b_i,v_i)}{\phi_i(b_i,v_i)} = \begin{cases} \gamma[\beta_j(v_i) - b_i] + (1-\gamma) \int\limits_{\min(\beta_j^{-1}(b_i))}^{\bar{v}} [\beta_j(v_j) - b_i] f(v_j) dv_j & \text{if} \quad b_i < \beta_j(v_i) \\ (1-\gamma) \int\limits_{\beta_j^{-1}(b_i)}^{\bar{v}} [\beta_j(v_j) - b_i] f(v_j) dv_j & \text{if} \quad b_i \geq \beta_j(v_i) \end{cases}$$

Then the best-response of bidder i is to bid zero when $v_i \leq \gamma\beta_j(v_i) + (1-\gamma)E[\beta_j(v_j)]$ and bid a positive amount for larger v_i. In the symmetric equilibrium, $\beta(v) = 0$ for $v \leq \hat{v}$ and $\beta(v) = \tilde{\beta}(v)$ for $v > \hat{v}$ where $\hat{v} = (1-\gamma)E[\beta(v)]$ and $v - \tilde{\beta}(v) \equiv (1-\gamma)\left[\int_v^{\bar{v}} \tilde{\beta}(v_j) f(v_j) dv_j - \tilde{\beta}(v)(1 - F(v))\right]$ for all $v \in [\hat{v}, \bar{v}]$ which is equivalent to (3.2). **(b):** For the nearest-VCG rule, $\frac{MP_i(b_i,v_i)}{\phi_i(b_i,v_i)} = \frac{\gamma}{2}\beta_j(v_i) + \frac{1-\gamma}{2}\int_0^{\bar{v}} \beta_j(v_j) f(v_j) dv_j$. Then the best-response of bidder i is to bid zero when $v_i \leq 0.5[\gamma\beta_j(v_i) + (1-\gamma)E[\beta_j(v_j)]]$ and bid a positive amount for larger v_i. In the symmetric equilibrium, $\beta(v) = 0$ for $v \leq \hat{v}$ and $\beta(v) = \tilde{\beta}(v)$ for $v > \hat{v}$ where $\hat{v} = 0.5(1-\gamma)E[\beta(v)]$ and $v - \tilde{\beta}(v) \equiv \frac{1}{2}\left[\gamma\tilde{\beta}(v) + (1-\gamma)E[\tilde{\beta}(v_j)]\right]$ for all $v \in [\hat{v}, \bar{v}]$. It follows that $\tilde{\beta}'(v) = \frac{2}{2+\gamma}$ and $\tilde{\beta}(v) = \frac{2}{2+\gamma}(v - \hat{v})$ on the interval $[\hat{v}, \bar{v}]$ where $\hat{v} = \frac{1-\gamma}{2+\gamma}\int_{\hat{v}}^{\bar{v}} (v_j - \hat{v}) f(v_j) dv_j$ which is equivalent to (3.4). The equation has a unique solution $\hat{v}$ on $(0, \bar{v})$ which is strictly decreasing with γ. **(c):** For the nearest-bid rule,

$$\frac{MP_i(b_i,v_i)}{\phi_i(b_i,v_i)} = \begin{cases} \gamma b_i + (1-\gamma)\left[\int\limits_0^{\beta_j^{-1}(b_i)} \beta_j(v_j)f(v_j)dv_j + \int\limits_{\beta_j^{-1}(b_i)}^{\bar{v}} b_i f(v_j)dv_j\right] & \text{if} \quad b_i < \beta_j(v_i) \\ \gamma\beta_j(v_i) + (1-\gamma)\left[\int\limits_0^{\beta_j^{-1}(b_i)} \beta_j(v_j)f(v_j)dv_j + \int\limits_{\beta_j^{-1}(b_i)}^{\bar{v}} b_i f(v_j)dv_j\right] & \text{if} \quad b_i \geq \beta_j(v_i) \end{cases}$$

Then the best response of bidder i is to bid $b_i > 0$ for any $v_i \in (0,\bar{v})$. In the symmetric equilibrium, $v - \beta(v) \equiv \gamma\beta(v) + (1-\gamma)\left[\int_0^v \beta(v_j)f(v_j)dv_j + \beta(v)(1-F(v))\right]$ for all $v \in [0,\bar{v}]$ which is equivalent to (3.5). □

PROOF OF COROLLARY 2. **(a):** For the proxy rule, $\beta_\gamma(v) \leq \beta_{\gamma'}(v)$ for all $v \in [0,\bar{v}]$ follows since $\tilde{\beta}_\gamma(\bar{v}) = \tilde{\beta}_{\gamma'}(\bar{v})$ and $\tilde{\beta}'_\gamma(v) \geq \tilde{\beta}'_{\gamma'}(v)$. **(b):** For the nearest-VCG rule, $\hat{v}(\gamma)$ is strictly decreasing function. Due to linearity of the bidding function with slope $\frac{2}{2+\gamma}$, $\beta_\gamma(v) \leq \beta_{\gamma'}(v)$ for $v \in [0,\tilde{v}]$ and $\beta_\gamma(v) \geq \beta_{\gamma'}(v)$ for $v \in [\tilde{v},\bar{v}]$ where $\tilde{v} \in (\hat{v}(\gamma),\bar{v}]$. If $\tilde{v} = \bar{v}$, then $MP_i^\gamma(b_i,\bar{v}) < MP_i^{\gamma'}(b_i,\bar{v})$ and the best-response of bidder i with $v_i = \bar{v}$ should drop under γ' which is a contradiction. It follows that $\tilde{v} \in (\hat{v}(\gamma),\bar{v})$. **(c):** For the nearest-bid rule, $\beta_\gamma(v) \geq \beta_{\gamma'}(v)$ for all $v \in [0,\bar{v}]$ follows since $\beta_\gamma(0) = \beta_{\gamma'}(0)$ and $\beta'_\gamma(v) \geq \beta'_{\gamma'}(v)$. □

PROOF OF COROLLARY 3. **(a):** For the proxy rule, $\beta_{F_1}(v) \leq \beta_{F_2}(v)$ for all $v \in [0,\bar{v}]$ follows since $\tilde{\beta}_{F_1}(\bar{v}) = \tilde{\beta}_{F_2}(\bar{v})$ and $\tilde{\beta}'_{F_1}(v) \geq \tilde{\beta}'_{F_2}(v)$. **(b):** For the nearest-VCG rule, $\beta_{F_1}(v) \leq \beta_{F_2}(v)$ for all $v \in [0,\bar{v}]$ follows since $\hat{v}_{F_2} \leq \hat{v}_{F_1}$. **(c):** For the nearest-bid rule, $\beta_{F_1}(v) \leq \beta_{F_2}(v)$ for all $v \in [0,\bar{v}]$ follows since $\beta_{F_1}(0) = \beta_{F_2}(0)$ and $\beta'_{F_1}(v) \leq \beta'_{F_2}(v)$. □

PROOF OF PROPOSITION 2. For the proxy rule, the expected marginal payment of bidder i is $MP_i(b_i,v) = G(\beta_j(v)+b_i) - G(2b_i)$ when $b_i \leq \beta_j(v)$ and $MP_i(b_i,v) = 0$ when $b_i > \beta_j(v)$. If a symmetric equilibrium exists, $MP_i(\beta(v_i),v_i) = 0$. But then $\beta(v) = v$ by Proposition 1 since $\beta(v) > 0$ for any $v > 0$. □

PROOF OF PROPOSITION 3. In a symmetric equilibrium, $\beta(v) > 0$ for any $v > 0$. Then $(v_i - \beta(v_i))g(2\beta(v_i)) = MP_i(\beta(v_i),v_i)$ for all pricing rules. Then (4.4) follows since $MP_i^{Proxy}(\beta(v_i),v_i) = 0$, strictly less than $MP_i^{N-VCG}(\beta(v_i),v_i) = 1/2[G(2\beta(v)) - G(\beta(v))]$ which is in turn strictly less than $MP_i^{N-BID}(\beta(v_i),v_i) = 1/2[G(2\beta(v))]$. □

PROOF OF THEOREM 2. By Proposition 1, the first-order conditions are given by (2.3). For all pricing rules, $\phi_i(b_i,v_i) = g(b_i + \beta(v_i))$. In a symmetric equilibrium, $\beta(v) > 0$ for any $v > 0$. **(a):** For the proxy rule, the expected marginal payment of bidder i is $MP_i(b_i,v) = G(\beta_j(v)+b_i) - G(2b_i)$ when $b_i \leq \beta_j(v)$ and $MP_i(b_i,v) = 0$ when $b_i > \beta_j(v)$. In a symmetric equilibrium $\beta(v) = v$ by Proposition 2. In an asymmetric equilibrium where $\beta_i(v) < \beta_j(v)$, bider j must be bidding truthfully. If $\beta_j(v) = v$, then the best response of bidder i is to bid $b_i = v$ since $(v-b_i)\,g(v+b_i) \geq G(v+b_i) - G(2b_i)$ (with a strict sign for all $b_i < v$) for all $b_i \in [0,v]$. The last inequality follows for $G(u) = (u/\bar{u})^\sigma$ since it is equivalent to $h(x) = x^\sigma + \sigma(1-x) \geq 1$ where $x = \frac{2b_i}{v+b_i} \in [0,1]$. It is satisfied if and only if $\sigma > 1$ since $h(0) = \sigma$, $h(1) = 1$ and $h'(x) < 0$ for all $x \in [0,1)$. **(b):** For the nearest-VCG rule, the expected marginal payment of bidder i is given by $MP_i(b_i, v) = \frac{1}{2}\big[G(\beta_j(v) + b_i) - G(b_i)\big]$. A fully asymmetric equilibrium cannot exist for $\sigma > 1$ since $2vg(v) > G(v)$. For a partially asymmetric equilibrium where $b_j(v) = x\,\beta_i(v)$ with $x \in (0,1)$ and $G(u) = (u/\bar{u})^\sigma$, according to the first-order conditions (2.3),

$$\beta_i(v) \;=\; \frac{2\,\sigma}{2\sigma + 1 + x - (1+x)^{(1-\sigma)}}\, v \qquad \beta_j(v) \;=\; \frac{2\,\sigma\,x}{2\sigma + 1 + x - (1+x)^{(1-\sigma)}}\, v$$

where $2\,\sigma\,(1-x) \;=\; (1+x)^{(1-\sigma)}(1 - x^\sigma)$. It can be shown that the above system of equations does not admit solutions such that $x < 1$ for $\sigma \geq 1$. The symmetric equilibrium is given by $\beta(v) = \frac{\sigma}{1+\sigma-2^{-\sigma}}\, v$ (plug $x = 1$). This equilibrium exists when $2\sigma(v - b_i) \geq \Psi(b_i)$ for all $b_i \in [0, \beta(v)]$ and $2\sigma(v - b_i) \leq \Psi(b_i)$ for all $b_i \in [\beta(v), v]$ where $\Psi(b_i) = (b_i + \beta(v)) - \frac{b_i^\sigma}{(\beta(v)+b_i)^{\sigma-1}}$. For $\sigma \geq 1$, $\Psi''(b_i) \leq 0$. Then $\Psi'(b_i)$ is a decreasing function that is positive at $b_i = v$, implying that $\Psi'(b_i) \geq 0$ for all $b_i \in [0, v]$. Then $\Psi(b_i)$ is increasing on $[0, v]$ and the symmetric equilibrium exists. **(c):** For the nearest-bid rule, the expected marginal payment of bidder i is given by

$$MP_i(b_i, v) = \begin{cases} \frac{1}{2}\big[G(\beta_j(v) + b_i) - G(\beta_j(v) - b_i)\big] & \text{if} \quad 0 \leq b_i \leq \beta_j(v) \\ \frac{1}{2}\big[G(\beta_j(v) + b_i) - G(b_i - \beta_j(v))\big] & \text{if} \quad \beta_j(v) < b_i \end{cases}$$

Note that $b_i = 0$ is never a best response when $v > 0$ since then $MP_i(0, v) = 0$ and $vg(\beta(v) + v) > 0$. Now suppose that $0 < \beta_j(v) < \beta_i(v)$. Then $MP_i(\beta_i(v), v) = MP_j(\beta_j(v), v)$ and $\beta_j(v) = \beta_i(v)$ by first-order conditions (2.3). Thus, there are no asymmetric equilibria for this payment rule. In a symmetric equilibrium $b_i = \beta(v)$ and $MP_i(\beta(v_i), v) = G(2\beta(v))/2$. Then, for $G(u) = (u/\bar{u})^\sigma$, according to the first-order conditions (2.3), $\beta(v) \;=\; \frac{\sigma}{1+\sigma}\, v$. This equilibrium exists when $2\sigma(v - b_i) \geq \Psi(b_i)$ for all $b_i \in [0, \beta(v)]$ and $2\sigma(v - b_i) \leq \Psi(b_i)$ for all $b_i \in [\beta(v), v]$ where $\Psi(b_i) = (b_i + \beta(v)) - (\beta(v) - b_i)\left[\frac{\beta(v)-b_i}{\beta(v)+b_i}\right]^{\sigma-1}$. The first inequality is satisfied for any $\sigma > 0$ since the left-hand side is strictly decreasing, the right-hand side is strictly increasing and the inequality is still satisfied at $b_i = \beta(v)$. The second inequality is satisfied for $\sigma \geq 1$. □

References

[1] Ausubel, Lawrence M., Christina Aperjis and Oleg Baranov. 2018. "Market Design and the FCC Incentive Auction," *working paper*.

[2] Ausubel, Lawrence M. and Oleg Baranov. 2017. "A Practical Guide to the Combinatorial Clock Auction." *Economic Journal* 127(605): 334-350.

[3] Ausubel, Lawrence M. and Paul Milgrom. 2002. "Ascending Auctions with Package Bidding." *Frontiers of Theoretical Economics* 1(1): 1-42.

[4] Ausubel, Lawrence M. and Paul Milgrom. 2006. "The Lovely but Lonely Vickrey Auction." *in (P. Cramton, Y. Shoham and R. Steinberg, eds.), Combinatorial Auctions*, Cambridge: MIT Press, 17-40.

[5] Clarke, Edward H. 1971. "Multipart Pricing of Public Goods." *Public Choice* 11: 17-33.

[6] Crémer, Jacques and Richard P. McLean. 1985. "Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist when Demands are Interdependent." *Econometrica* 53(2): 345-361.

[7] Day, Robert W. and Peter Cramton. 2012. "The Quadratic Core-Selecting Payment Rule for Combinatorial Auctions." *Operations Research* 60(3): 588-603.

[8] Day, Robert W. and Paul Milgrom. 2008. "Core-selecting Package Auctions." *International Journal of Game Theory* 36(3-4): 393-407.

[9] Day, Robert W. and S. Raghavan. 2007. "Fair Payments for Efficient Allocations in Public Sector Combinatorial Auctions." *Management Science* 53(9): 1389-1406.

[10] Erdil, Aytek and Paul Klemperer. 2010. "A New Payment Rule for Core-Selecting Package Auctions." *Journal of the European Economic Association* 8(2-3): 537-547.

[11] Goeree, Jacob K. and Yuanchuan Lien. 2016. "On the Impossibility of Core-Selecting Auctions." *Theoretical Economics* 11(1): 41-52.

[12] Groves, Theodore. 1973. "Incentives in Teams." *Econometrica* 41(4): 617-31.

[13] Hafalir, Isa E. and Hadi Yektaş. 2015. "Core Deviation Minimizing Auctions." *International Journal of Game Theory* 44(2): 367-376.

[14] Krishna, Vijay and Robert Rosenthal. 1996. "Simultaneous Auctions with Synergies." *Games and Economic Behavior* 17(1): 1-31.

[15] McAfee, Preston R. and Philip J. Reny. 1992. "Correlated Information and Mechanism Design." *Econometrica* 60(2): 395-421.

[16] Ott, Marion, and Marissa Beck. 2013. "Incentives for Overbidding in Minimum-Revenue Core- Selecting Auctions." *RWTH Aachen University*. Available at http://hdl.handle.net/10419/79946.

[17] Sano, Ryuji. 2010. "An Equilibrium Analysis of a Package Auction with Single-Minded Bidders." Available at SSRN: http://ssrn.com/abstract=1594471.

[18] Sano, Ryuji. 2012. "Non-bidding Equilibrium in an Ascending Core-Selecting Auction." *Games and Economic Behavior* 74(1): 637-650.

[19] Vickrey, William. 1961. "Counterspeculation, Auctions, and Competitive Sealed Tenders." *Journal of Finance* 16(1): 8-37.

[20] Wilson, Robert. 1987. "Game-Theoretic Analyses of Trading Processes." in *Advances in Economic Theory: Fifth World Congress*, (Truman F. Bewley ed.), Cambridge: Cambridge University Press, 33-70.