Revealed Preference and Activity Rules in Dynamic Auctions*

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Activity rules—constraints that limit bidding in future rounds based on past bids—are intended to limit strategic bidding delays in high-stakes auctions. This article provides a general treatment of activity rules. Traditional point-based rules are effective for homogeneous goods and reasonably suited for substitute goods. However, they are simultaneously too strong and too weak for general environments; they allow parking, while sometimes preventing straightforward bidding. We prove that the activity rule operationalizing the generalized axiom of revealed preference (GARP) is essentially the unique rule that enforces the Law of Demand while enabling straightforward bidding and never producing "dead ends".

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1 Introduction

Dynamic auctions, in which a set of related items are offered simultaneously for iterative bidding, are commonly used today for allocating spectrum, electricity, natural gas, offshore wind energy leases, emission reduction incentives, diamonds and other natural resources. Such dynamic market designs facilitate the "discovery" of competitive prices and the efficient allocation of the items. At the same time, these objectives are likely to be undermined to the extent that bidders can employ a strategy of delaying submission of meaningful bids until late in the auction—described evocatively in the literature as a "snake in the grass" strategy.

An *activity rule* is a constraint on the bids that a bidder is permitted to submit in a given round of a dynamic auction, as a function of the bidder's prior bidding history. Activity rules have been among the key innovations in dynamic auctions, in that they restrain strategic bidding delays and thereby enable the auctions to accomplish their objectives. The first known use of an activity rule occurred in the early US Federal Communications Commission (FCC) auctions. In the proceedings immediately prior to FCC Auction No. 1, the Commission provided the following rationale for activity rules:

> *In order to ensure that simultaneous auctions with simultaneous stopping rules close within a reasonable period of time and to increase the information conveyed by bid prices during the auction, we believe that it is necessary to impose an activity rule to prevent bidders from waiting until the end of the auction before participating. Because simultaneous stopping rules generally keep all licenses open for bidding as long as anyone wishes to bid, they also create an incentive for bidders to hold back until prices approach equilibrium before making a bid. As noted above, this could lead to very long auctions. Delaying serious bidding until late in the auction also reduces the information content of prices during the course of an auction. Without an activity rule, bidders cannot know whether a low level of bidding on a license means that the license price is near its final level or if instead many serious bidders are holding back and may bid up the price later in the auction.*[2]

The consensus view of the subsequent economics literature has been that an effective activity rule tends to promote price discovery and transparency, as well as to facilitate rapid convergence to equilibrium.[3]

Even in settings with pure private values—which is a maintained assumption until Section 5 of this paper—there are any number of reasons why bidders in dynamic auctions would hold

[2] Federal Communications Commission (1994), Implementation of Section 309(j) of the Communications Act - Competitive Bidding, Fifth Report and Order, 9 FCC Rcd. 5532, `https://docs.fcc.gov/public/attachments/FCC-94-178A1.pdf`.

[3] See, for example, McMillan (1994, pp. 153–155), McAfee and McMillan (1996, pp. 160–161), Milgrom (2000, pp. 247–249 and pp. 258–261), Milgrom (2004, pp. 267–268), Ausubel and Cramton (2004, p. 487) and Ausubel, Cramton and Milgrom (2006, p. 119).

back on their bids, absent an activity rule. First, observe that bids must be treated as binding commitments (or else there would be endless opportunities to game the auction) and, as a result, auction rules typically impose significant withdrawal penalties. The most direct way for bidders to escape incurring withdrawal penalties is to delay placing their bids for as long as possible. This issue can arise in any dynamic auction, since bidders are exposed both to secular changes in market fundamentals[4] and to idiosyncratic changes in their own objectives.[5] It becomes particularly important in spectrum auctions that do not include package bidding, as spectrum licenses are often complementary and bidders are exposed to the risk of winning some licenses but not the complementary licenses. Second, bidders in high-stakes auctions are often subject to budget constraints.[6] Bidders may therefore want to avoid tying up their limited budgets until late in an auction, and to try to take advantage of bargains that arise when their opponents' budgets are tied up early.[7] Third, an incumbent firm that faces entrants in an auction for an essential input such as spectrum may have specific incentives to draw out the length of the auction as long as possible, so as to delay the day when entrants can enter. Fourth, a large bidder may have incentives to extend an auction in order to impose costs on its smaller rivals, since higher participation costs falling on all bidders may have a disproportionate impact on smaller competitors. Fifth, late bidding may represent a form of tacit collusion: bidders bid late in order to avoid triggering bidding wars.[8] Finally, in an environment where behavioral considerations hold sway, bidders may similarly bid late in order to avoid provoking rivals to respond in the heat of the auction.[9]

In environments with interdependent values, i.e., when a bidder's value depends on its

[4] A particularly striking example of a change in market fundamentals occurred in the UK 3G spectrum auction, which extended from March 6 to April 27, 2000. The NASDAQ composite index reached its intra-day peak of 5132.52 during the first week of this auction; this price level would not again be reached until June 2015. By the seventh week of this auction, the NASDAQ index had already dropped as low as 3227.04—or 37% off its peak. A second example occurred in the first Electricité de France (EDF) Generating Capacity Auction, a two-day auction with the inauspicious starting date of September 11, 2001.

[5] In the FCC's recent Broadcast Incentive Auction, AT&T was initially the most aggressive bidder in the auction and, in round 21, it sought \$7.5 billion in licenses. However, beginning in round 22, the company suddenly and massively curtailed its bidding, ultimately winning less than \$1 billion in licenses despite minimal subsequent price increases. Observers variously attributed AT&T's abrupt change of heart to receiving positive indications on the FirstNet public safety procurement, which provided an alternative source of low-frequency spectrum, and to its merger deal with Time Warner, which was negotiated during the auction and provided an alternative use of financial resources.

[6] See, for example, Pitchik and Schotter (1988) and Che and Gale (1998).

[7] This explanation is modeled in Milgrom (2000, pp. 258-261).

[8] This argument is provided by Roth and Ockenfels (2002) as a reason for bid sniping on eBay.

[9] "There are many reasons to snipe. A lot of people that bid on an item will actually bid again if they find they have been outbid, which can quickly lead to a bidding war. End result? Someone probably paid more than they had to for that item. By sniping, you can avoid bid wars." (As quoted from the website esnipe.com by Roth and Ockenfels, 2002, footnote 9.) Note that this explanation is very difficult to distinguish empirically from the fifth explanation.

opponents' information as well as its own, a high bid by one bidder may signal to other bidders that a particular item is worth more than they originally thought. Obviously then, bidders have additional reasons, above and beyond the private values case, to delay bidding on items of interest until the last possible moment.[10]

Observe that an important characteristic of many applications of these modern dynamic auctions is that they involve heterogeneous goods (e.g., spectrum auctions typically include licenses for different geographic areas or different spectrum bands, while diamond auctions invariably include stones of different size, color and clarity). To address this heterogeneity, FCC Auctions 1, 3 and 4—and the vast majority of subsequent multi-round spectrum auctions conducted worldwide to date—have utilized *point-monotonic* activity rules.[11] For example, a New York City license might be assigned 100 points, a Los Angeles license might be assigned 60 points, and a Washington DC license might be assigned 30 points. In each round of the auction, the bidder's activity is defined to be the sum of the points of all of the licenses on which the bidder was active. The activity rule essentially requires the bidder's activity to be (weakly) decreasing from round to round.[12] As such, a bidder could bid initially for NY only (100 points), then switch to the $\{LA, DC\}$ combination (90 points), then bid for LA only (60 points), and finally bid for DC only (30 points). However, the activity rule prevents the bidder from switching in the opposite direction.

Unfortunately, point-monotonic activity rules give rise to a number of problems. Bidders discovered in early FCC auctions that some licenses inevitably would be assigned disproportionately many points relative to their value and activity could be stockpiled there for later use. This phenomenon became so prevalent that it was promptly given a name: "parking". Meanwhile, it has been shown more recently that the difficulty is not merely a miscalibration of points. There exist valuation profiles and price histories such that point monotonicity would prevent bidders from bidding their true demands for *any* choice of points—and if bidders attempted to bid their true demands, the outcome would necessarily be inefficient.[13] Not only is the process of assigning points subject to intense stakeholder lobbying, but it is inherently doomed to failure.

[10] This explanation is described, for example, in footnote 4 of Ausubel and Cramton (2004).

[11] FCC Auction No. 2 was conducted as a sequence of traditional open-outcry auctions, each for a single license, and it did not utilize an activity rule.

[12] More precisely, in FCC Auction No. 4, the activity rule provided for three stages with increasing activity requirements in each stage. In the first stage, a bidder wishing to maintain its current eligibility was required to be active ("active" meaning submitting a new bid or maintaining a standing high bid) on licenses encompassing at least one-third of the points for which it was eligible in that particular round. In the second stage, the minimum activity requirement was raised from one-third to two-thirds. In the third stage, a bidder wishing to maintain its current eligibility was required to be active on licenses encompassing 100 percent of the points for which it was eligible in that particular round, i.e., point monotonicity. *Procedures, Terms and Conditions of Auction*, pp. 10–11, contained in Federal Communications Commission (1994).

[13] The first proof of this result was in Daniel Hauser's (2012) University of Maryland undergraduate honors thesis. For completeness, we provide a somewhat simpler argument of Hauser's result in Section 3.

In this paper, we characterize point-monotonic activity rules as "simultaneously too weak and too strong."[14]

Thus, the specification of the "right" activity rule is a nontrivial and important problem in auction design. There is a clear answer for ascending auctions of a single item: a rule of irrevocable exit (Milgrom and Weber, 1982, p. 1104). There is also a clear answer for ascending auctions of a homogeneous good: a requirement of (weakly) decreasing demands (Ausubel, 2004, p. 1460). The focus of the current article is the specification of the "right" activity rule for auctions of heterogeneous goods.

As seen in the second paragraph, the earliest discussions of activity rules were framed largely in terms of assuring that auctions "close within a reasonable period of time". The reason for this emphasis was that most early spectrum auctions employed the simultaneous multiple round auction (SMRA) format. Under the simultaneous closing rule, an SMRA stays open for *all* items until a round elapses in which no new bids are submitted for *any* item. Consequently, even though spectrum auctions do not have fixed ending times—unlike eBay auctions, which are the focus of the bid sniping literature—every bidder possesses the unilateral means to guarantee that the current round is not the final round, simply by submitting at least one new bid. Without an activity rule, the SMRA can fail catastrophically merely by lasting for thousands of rounds.

More recent discussions of activity rules have arisen in the context of second-generation auction designs such as the combinatorial clock auction (CCA) and the clock auction, which have begun to supplant the SMRA in practical applications. In these newer designs, the simultaneous closing rule takes on a somewhat different meaning: the auction stays open for all items until a round elapses without excess demand for any item. Individual bidders can no longer unilaterally guarantee that the auction will continue beyond a given round, since their opponents' demands could evaporate. As such, there is somewhat less concern about the duration of the auction and greater interest in more nuanced issues related to truthful bidding. For example, Ausubel, Cramton and Milgrom (2006) proposed a new activity rule that operationalized the weak axiom of revealed preference (WARP), in the same paper in which they proposed the CCA. Subsequently, Harsha, Barnhart, Parkes and Zhang (2010) and Ausubel and Baranov (2014) proposed activity rules that operationalize the generalized axiom of revealed preference (GARP), both in the context of the CCA and related combinatorial auctions.

In the current article, we introduce a series of axioms that formalize desirable properties in activity rules and we characterize the rules that satisfy these axioms. Our first axiom gets to the heart of activity rules. An activity rule should inhibit delayed bidding, which includes feigning lack of interest in target items early in the auction at low prices, only to express interest later

[14] More precisely, point monotonicity is too weak in the sense that it fails Axiom 1 (an activity rule should enforce the Law of Demand), and it is too strong in the sense that it fails Axiom 2 (an activity rule should permit straightforward bidding).

in the auction at higher prices. A minimal requirement is thus for the activity rule to enforce the Law of Demand, meaning that demand is weakly decreasing in prices.

Our second axiom is that the activity rule should permit straightforward bidding. We provide two versions of this: Axiom 2, which holds that the bidder's entire demand correspondence is feasible; and Axiom 2$'$, which requires the feasibility of at least one element of the demand correspondence. A key justification for these axioms is that efficiency is a primary objective in auction design. It would seem problematic to try to put items in the hands of the bidders who value them the most using a mechanism that prevented bidders from expressing their true demands at current prices. Moreover, one of the canons of the mechanism design literature is strategyproofness; clearly, an auction mechanism would run counter to the received wisdom if its activity rule precluded truthful bidding.

Observe that Axiom 1 can prevent a bidder from re-bidding its most recent demand, when relative prices change. (This is never the case with point monotonicity, which always allows a bidder to reaffirm its most recent bid.) Our third axiom, referred to as *no dead ends*, seems like a minimal requirement to protect bidders during the bidding process. While the bidder may not necessarily be allowed to reaffirm its last bid, the bidder should always have the option of bidding for one of the bundles that it previously demanded in the auction. Especially when prices are monotonically increasing, this provides the bidder with some degree of protection during the bidding process.

Our main results are that these three axioms cause rejection of the WARP activity rule, but lead inexorably toward GARP or SARP activity rules. In particular, while the WARP activity rule satisfies Axioms 1 and 2, it fails to satisfy Axiom 3 (Theorem 1). The GARP rule satisfies all three axioms and, with nonlinear bundle prices, it is the unique activity rule that satisfies all three axioms (Theorems 3 and 5). Furthermore, the GARP rule exhibits the desirable property of *payoff continuity*: a mistake that costs a bidder δ in payoff in the current round can never reduce the bidder's payoff by more than δ in any future round (Theorem 4). Meanwhile, the strong axiom of revealed preference (SARP) rule is the most restrictive activity rule that satisfies Axioms 1, 2$'$ and 3 (Theorem 7). And, since the literature posits that an important role of dynamic auctions is to allow the updating of valuations, we define a notion of "price-justified updating" and prove that the GARP rule allows it (Theorem 8).

We also discuss four relaxations of the GARP activity rule, which accommodate (1) auction formats with provisional winners, (2) bidders with budget constraints, (3) bounded rationality, and (4) a hybrid of the point monotonicity and GARP approaches. Finally, we examine briefly some empirical evidence from four recent spectrum auctions, as proof of concept for the novel activity rules.

Employing activity rules that are as strict as possible (while still satisfying our axioms) takes on added importance in auction formats such as the CCA where pricing is based on expressed opportunity cost. Unconstrained, bidders might misrepresent their demands so as to impose higher costs on their rivals. Conversely, with the tight constraints that come from the GARP activity rule, it is possible to replace rivals' actual bids with upper bounds on their allowable bids in the opportunity-cost calculation, as we propose in a companion paper, Ausubel and Baranov (2019).

One literature related to this paper is the fairly short set of articles that develop or explain activity rules. These have already been surveyed in the Introduction up to this point. A second related literature is the older and more voluminous literature on revealed preference. Owing to its volume, our review shall be quite abbreviated, limiting attention to those articles that most directly relate to our paper. The reader is directed to an excellent survey by Varian (2006), on which our review is partly based. Samuelson (1938) introduced the notion of revealed preference and formulated the condition that has become known as WARP. Houthakker (1950) proved how one could use the revealed preference relation to construct a set of indifference curves for a general set of goods and, in the process, introduced the stricter condition that has become known as SARP. Afriat (1967) asked the question: When can we rationalize a finite set of observed prices and demand choices by a utility function? His answer required formulating GARP, which is only a slight generalization of SARP. Afriat's profound result is that a finite set of observations can be rationalized if and only if it satisfies GARP. The GARP condition proves especially convenient when demand correspondences may be multiple valued. Warshall (1962) showed that the transitive closure computation needed to test for GARP can be completed in polynomial time. Rochet (1987) treated the special case of quasilinear GARP that we use in this paper and proved the counterpart to Afriat's theorem.

The remainder of our article is organized as follows. Section 2 introduces the foundations of our modeling, including the three axioms for activity rules. Section 3 analyzes monotonicity-based activity rules, including point monotonicity. Section 4 contains our main results—characterizations of the WARP, GARP and SARP rules. Section 5 addresses the updating of valuations and proves that the GARP rule allows "price-justified updating". Section 6 formulates four relaxations of the GARP activity rule. Section 7 examines briefly some empirical evidence from four recent spectrum auctions. Section 8 concludes. Many of the proofs are relegated to the Appendix.

2 Preliminaries

The Model

In this article, we study activity rules in dynamic auctions that operationalize the classical notion of Walrasian tâtonnement (Walras, 1874). An auctioneer seeks to sell M types of heterogeneous indivisible goods to a set of bidders. In each successive round, the auctioneer names prices, bidders respond with quantities, and the auctioneer adjusts prices seeking to bring demand and supply into balance. The theoretical foundation for using this procedure is given by the First Fundamental Welfare Theorem: if bidders who report their demand truthfully trade to a competitive equilibrium, then the outcome is Pareto optimal. However, we leave open the exact details of the auction process. It could, for example, be a standard clock auction in which bidders win their final demands and pay the dot product of the final prices and their final demands. Alternatively, it could be a combinatorial clock auction (CCA) or another iterative version of the VCG mechanism.

More specifically, the supply of M heterogeneous indivisible goods is denoted by $S = (s(1), ..., s(M)) \in \mathbb{Z}^M_{++}$. The set of all possible bundles of goods is denoted by $\Omega = \{ (z(1), ..., z(M)) \in \mathbb{Z}^M_{+} : \ 0 \leq z(m) \leq s(m), \ \ \forall m = 1, ..., M\}$. The auctioneer wishes to allocate these goods to a set of bidders $N = \{1, \ldots, n\}$. For each bidder $i \in N$, the bidder's preferences over packages in Ω are characterized by a value function $v_i(\cdot)$. The value of the zero bundle, $\vec{0}$, is normalized to zero. We make the following assumptions about value functions:

(A1) Pure Private Values: Each bidder i knows its own value for any bundle of goods and its value is not affected by the values of other bidders;

(A2) Quasilinear Preferences: Each bidder i's payoff from winning bundle z and making a payment of y is given by $v_i(z) - y$.

The set of all value functions that satisfy assumptions (A1) and (A2) will be denoted by V. Monotonicity of value functions (i.e., free disposal) is not required for our main results,[15] but it will be assumed for an auxiliary result in Section 6. In some parts of the paper, assumptions (A1) and (A2) are relaxed to allow for interdependent values (see Section 5) and budget constraints (see Section 6).

In each round $t = 1, 2, ...$, the auctioneer announces a price vector $p^t \in \mathbb{R}^M_+$, and each bidder i responds with a demand vector $x_i^t \in \Omega$. After collecting demands in round t, the auctioneer decides whether to continue with the next round $t+1$ and solicit bidders' demands at a new price vector p^{t+1} or to conclude this phase of the auction. Without loss of generality, we assume that the new price vector never equals any of the prior price vectors, i.e., $p^{t+1} \neq p^s$ for any $s \leq t$. In some parts of the paper, we also assume that the clock prices are non-decreasing, i.e., $p^{t+1} \geq p^t$ for each t.

Let $D(p, v) \subseteq \Omega$ denote the demand correspondence of a bidder with value function $v(\cdot) \in V$ at prices $p \in \mathbb{R}^M_+$, i.e.,

$$D(p, v) = \underset{z \in \Omega}{\arg\max} \ \{ v(z) - p \cdot z \}. \tag{2.1}$$

[15] Observe that, even if bidders' value functions are non-monotonic, their preferences are locally nonsatiated through the assumed quasilinearity. This is because the preference relation is defined on bundles of goods *and money*, so at any point in the relevant domain, there is a nearby point (with ϵ more money) that is strictly preferred. Hence, the First Fundamental Welfare Theorem (e.g., Mas-Colell, Whinston and Green, 1995, Proposition 16.C.1) guarantees the Pareto optimality of any competitive equilibrium outcome.

Definition 1. *Bidder i is said to bid straightforwardly according to value function $v(\cdot)$ if its demand report $x_i^t \in D(p^t, v)$ in all rounds t. Bidder i is said to bid truthfully when it bids straightforwardly according to its true value function $v_i(\cdot)$.*

Axioms for Activity Rules

As motivated in the Introduction, we now introduce activity rules. Since activity rules constrain bidders based on their individual demand histories, we can omit bidder subscripts for the rest of the paper. We start by providing a formal definition of an activity rule. Let $X^t \subseteq \Omega$ denote the set of bundles that are feasible (i.e., available for bidding) for a given bidder in round t. For convenience, we assume that $X^1 = \Omega$.

Definition 2. *An activity rule is a constraint on the feasible set, X^t, of bids in each round $t \geq 2$ of the auction, as a function of the bidder's prior feasible bidding history $(p^1, x^1), ..., (p^{t-1}, x^{t-1})$.*

The next definition defines a partial order on the set of all activity rules:

Definition 3. *Activity rule A is weaker than activity rule B if, for any $t \geq 2$ and any bidding history $(p^1, x^1), ..., (p^t, x^t)$ that is feasible under rule B, the same bidding history is also feasible under rule A. Analogously, activity rule A is stricter than activity rule B if activity rule B is weaker than activity rule A.*

The primary purpose of the activity rule is to impede "snake in the grass" strategies by compelling bidders to foreshadow their true intentions early in the auction. In particular, activity rules should encourage serious bidding by making parking strategies as impractical as possible. The *Law of Demand*—a requirement that a bidder's demand and prices always move in opposite directions—is a minimal requirement with this flavor. Bidders are permitted to switch only to bundles that have become relatively less expensive.

Axiom 1. *An activity rule should enforce the Law of Demand:*[16] *for every round $t \geq 2$ and for every feasible bundle $z \in X^t$:*

$$(p^t - p^s) \cdot (z - x^s) \leq 0, \qquad \text{for all} \quad 1 \leq s < t. \tag{2.2}$$

At the same time, the activity rule would be too severe if, while inhibiting parking strategies, it also precluded truthful bidding. The designer wishes to retain truthful bidding because it (in combination with a Walrasian tâtonnement price adjustment process) is what potentially leads the auction process toward a Walrasian equilibrium and its desirable efficiency properties. Without truthful bidding, there can be no expectation of invoking the First Fundamental Welfare Theorem.

[16] Eq. (2.2) is Eq. (4.C.3) of Mas-Colell, Whinston and Green (1995), as restricted to quasilinear utility. Mas-Colell, Whinston and Green call this the *uncompensated law of demand (ULD)* property.

This is the motivation for Axioms 2 and 2′. Observe that, since the bidder's value function is private information, the activity rule needs to allow bidding according to *any* consistent set of objectives. We define two alternative notions (differing only in subtle ways) of the required truthful bidding:

Axiom 2. *An activity rule should allow straightforward bidding: for any value function* $v(\cdot) \in V$ *and for every round* $t \geq 2$*, a history of bidding straightforwardly according to* $v(\cdot)$ *in rounds* 1, ..., $t-1$ *implies that:*

$$D(p^t, v) \subseteq X^t. \tag{2.3}$$

Axiom 2′. *An activity rule should allow "limited" straightforward bidding: for any value function* $v(\cdot) \in V$ *and for every round* $t \geq 2$*, a history of bidding straightforwardly according to* $v(\cdot)$ *in rounds* 1, ..., $t-1$ *implies that:*

$$D(p^t, v) \cap X^t \neq \emptyset. \tag{2.4}$$

The only difference between these two formulations is that Axiom 2 assures the feasibility of *all* optimal demand bundles, while Axiom 2′ merely assures the feasibility of *at least one* optimal demand bundle.

Finally, the activity rule should never enable a malicious auctioneer to take advantage of a bidding mistake and shut a bidder out of the auction (or for a malicious bidder to take advantage of the situation and shut out its opponent). One way to implement this requirement, subsumed in essentially every activity rule that has ever been used in real-world dynamic auctions, is to allow a bidder to rebid its demand from the immediately previous round, i.e., $x^{t-1} \in X^t$ in each round t. While this provides a high degree of protection to the bidder, it is too strong an assumption for the general heterogeneous setting: first, it may violate the Law of Demand; and second, it does not generalize to non-monotonic price paths.[17] For present purposes, it suffices for the bidder to be able to rebid *some* prior demand, but not necessarily the bidder's most recent demand.

We define a *dead end* as a scenario in which no bundle that the bidder has demanded in any prior round remains feasible. In other words, the bidder is forced to demand a bundle (potentially the zero bundle) for which it has not shown interest at any price levels to date. Our third and final axiom requires no dead ends:

Axiom 3. *An activity rule should never produce a dead end: after any bidding history* $(p^1, x^1), ..., (p^{t-1}, x^{t-1})$*, the feasible set in round* $t \geq 2$ *must always include at least one of*

[17] For example, consider an auction of a homogeneous good in which $p^1 < ... < p^{t-1} < p^t$, but $p^{t+1} < p^{t-1}$. If $q^{t-1} > q^t$, then it would appear nonsensical to allow the bidder to rebid q^t at price p^{t+1}, given that the bidder has already revealed that it desires a quantity of at least q^{t-1} at this price. This could also be described as a violation of the Law of Demand.

the prior-demanded bundles, i.e.,

$$\{x^1, ..., x^{t-1}\} \cap X^t \neq \emptyset \; . \tag{2.5}$$

Axiom 3 is related to requiring continuity of the activity rule: a small change in the bidding history prior to round t should not lead to a large change in the set of feasible bundles in round t. This theme will be further developed in Theorem 4.

The reasoning in the proof of Theorem 3, below, proves that Axiom 2 implies Axiom 3 for the limited case of straightforward bidding; Axiom 2 allows a straightforward bidder to continue bidding straightforwardly, making it impossible to run into a dead end. Thus, the purpose of Axiom 3 is to protect non-straightforward bidders.

Observe that in a setting with homogeneous goods, Axiom 1 ensures straightforward bidding. Consequently, together with Axiom 2 and increasing prices, it implies the activity rule known as demand monotonicity. We have:

Definition 4 (Demand Monotonicity)**.** *With a homogeneous good* ($M = 1$)*, the feasible set of bids (quantities) in Round* $t \geq 2$ *is* $X^t = \{z \in \Omega : z \leq x^{t-1}\}$.

Proposition 1. *In an ascending auction of a homogeneous good* ($M = 1$)*, demand monotonicity is the unique activity rule that satisfies Axioms 1, 2 and 3 (with only a single item, demand monotonicity specializes to irrevocable exit).*

3 Monotonicity-Based Activity Rules

Throughout this section, we assume that the auctioneer uses non-decreasing clock prices since monotonic price paths are required for activity rules based on some form of demand monotonicity.

In homogeneous good settings, demand monotonicity is the unique activity rule satisfying all axioms (Proposition 1). With heterogeneous goods, there are two ways to extend the notion of demand monotonicity: (i) to require monotonicity of a bidder's demand in each good; or (ii) to require that a bidder's demand cannot include supersets of bundles that were demanded at lower prices. These constraints motivate strong and weak versions of the activity rule based on demand monotonicity.

Definition 5 (Strong Demand Monotonicity)**.** *The feasible set of bids in Round* $t \geq 2$ *is* $X^t = \{z \in \Omega : z(m) \leq x^{t-1}(m) \text{ for all } m = 1, ..., M\}$.

Definition 6 (Weak Demand Monotonicity)**.** *The feasible set of bids in Round* $t \geq 2$ *is* $X^t = \{z \in \Omega : \nexists\, s < t \text{ such that } z \geq x^s \text{ and } z \neq x^s\}$.

In general, strong demand monotonicity satisfies Axioms 1 and 3 but it is too restrictive to satisfy Axiom 2; in particular, this activity rule does not allow bidders to express substitution among goods. By contrast, weak demand monotonicity satisfies Axioms 2 and 3 but it is too weak to enforce Axiom 1. To illustrate, consider an example with two unique goods, A and B. Under strong demand monotonicity, a bidder is not allowed to switch its demand between A to B at any price; and under weak demand monotonicity, a bidder can switch between A to B no matter the price. As a result, both rules are too extreme to be relevant in practice.

A popular practical alternative, the point-monotonic activity rule, eliminates some of the deficiencies and combines desirable properties of strong and weak demand monotonicity rules. To implement it, the auctioneer needs to assign a specific number of "eligibility points" to every bundle in Ω. A standard approach is to assign eligibility points to each good and calculate the number of eligibility points associated with a bundle by summing eligibility points over all goods included in the bundle. Formally, let $e = (e(1), ..., e(M)) \in \mathbb{Z}^M_{++}$ denote a vector of eligibility points associated with goods in M and let $E(z) = e \cdot z$ denote the total number of eligibility points associated with any bundle $z \in \Omega$. The key idea behind point monotonicity is to require that the bidder's demand is monotonic in eligibility points.

Definition 7 (Point Monotonicity). *The feasible set of bids in Round $t \geq 2$ is $X^t = \{z \in \Omega : \ E(z) \ \leq \ E(x^{t-1})\}$.*

Point monotonicity is indeed weaker than strong demand monotonicity since it allows bidders to substitute goods within the current eligibility limit. It is also stricter than weak demand monotonicity as it eliminates violations of the Law of Demand that involve bidding for bundles that are larger in terms of eligibility points. However, the point-monotonic activity rule does not close the gap between the weak and strong notions of demand monotonicity—point monotonicity is still too weak to satisfy Axiom 1 and still too strong to satisfy Axiom 2.

First, point monotonicity is hopeless in terms of enforcing the Law of Demand (Axiom 1). For any choice of points, it allows a bidder to switch its demand (within its eligibility limit) independently of changes in prices and it can never stop a bidder from demanding a bundle that has become relatively more expensive.

Second, it is also impossible to accommodate straightforward bidding in a general setting (Axiom 2)—for any choice of points, there exist bidder values and a price trajectory rendering straightforward bidding impossible.[18] Consider a simple example with three unique goods, A, B and C, with the corresponding eligibility points $e(A)$, $e(B)$ and $e(C)$. On the one hand, a bidder might be interested in only one good and view all three goods as equally valuable, $v(A) = v(B) = v(C)$, requiring equal eligibility points $e(A) = e(B) = e(C)$ for unobstructed straightforward bidding under any non-decreasing price path. On the other hand, a bidder

[18]The first proof of this result is due to Daniel Hauser's (2012) University of Maryland senior thesis (Theorem 2).

might view good A and bundle BC as equally valuable, $v(A) = v(BC)$, and thus requiring that $e(A) = e(B) + e(C)$ for straightforward bidding.

There are several classes of value functions for which point monotonicity can accommodate straightforward bidding. Obviously, any choice of eligibility points enables straightforward bidding for additively-separable valuations (but so does the strong demand monotonicity rule). A more interesting example is due to Hatfield and Milgrom (2005) who showed that for value functions that satisfy the gross substitutes condition, a bidder's demand always satisfies the "law of aggregate demand"—the total *number* of demanded goods never goes up as prices increase. For such settings, straightforward bidding is enabled by assigning equal eligibility points to all goods. However, assigning equal points to goods with different values exacerbates the parking problem and is contrary to the common practice of setting eligibility points in proportion to estimated values (and spectrum licenses are anyway often regarded as complements).

To summarize this section, monotonicity-based activity rules are generally inconsistent with Axiom 1 and/or Axiom 2 while satisfying Axiom 3 (no dead ends). This is not surprising since Axioms 1 and 2 are both directly related to revealed preference and are sensitive to relative movements in prices and demands.

4 Revealed Preference Activity Rules

Unlike monotonicity-based activity rules, activity rules based on revealed preference do not require monotonic prices. For greater generality, we relax the assumption of monotonic price paths in this section.

Monotonicity-based activity rules are generally unable to enforce the Law of Demand (Axiom 1). The key distinction of activity rules based on revealed preference is the direct enforcement of the Law of Demand. In this section, we consider activity rules based on various versions of the revealed preference axioms: (i) the weak axiom of revealed preference (WARP); (ii) the generalized axiom of revealed preference (GARP); and (iii) the strong axiom of revealed preference (SARP).

While we impose quasilinearity, our exact statements of WARP and the Law of Demand are otherwise slightly weaker than is standard in the literature, in that: (i) demand is not required to be single-valued; and (ii) inequality (4.1) is not required to hold with strict inequality when $x^{t_1} \neq x^{t_2}$. Similarly, our statement of SARP in Section 4.3 does not require demand to be single-valued. The reason for our slightly weaker statements is that we are studying indivisible goods, so demands *must* be multiple-valued at some prices. Unlike with the standard definitions, where GARP does not imply WARP, our definitions ensure that SARP $\Longrightarrow$ GARP $\Longrightarrow$ WARP.

4.1 WARP Activity Rule

We start with a definition of WARP for quasilinear preferences. It is well known that the Law of Demand is equivalent to WARP under quasilinearity. We have:

Definition 8 (WARP)**.** *A collection of price-demand observations* $(p^k,\, x^k)$, $k = 1, ..., t$, *satisfies WARP for quasilinear preferences if for any choice of indices* $t_1, t_2 \in \{1, ..., t\}$,

$$p^{t_1} \cdot [x^{t_1} \; - \; x^{t_2}] \; + \; p^{t_2} \cdot [x^{t_2} \; - \; x^{t_1}] \; \leq \; 0. \tag{4.1}$$

The corresponding activity rule is defined as follows:

Definition 9 (WARP Activity Rule)**.** *The feasible set of bids in Round* $t \geq 2$ *is* $X^t = \{z \in \Omega : \; (p^1,\, x^1), ..., (p^{t-1},\, x^{t-1}), (p^t,\, z)$ *satisfies quasilinear WARP*$\}$.

By construction, the WARP activity rule satisfies Axiom 1. It also trivially satisfies Axiom 2 since straightforward bidding would never violate the Law of Demand. However, the WARP activity rule does not protect against dead ends. To illustrate, consider an example with three unique goods, A, B and C, and a bidding history provided in Table 1. It can be verified that bids made in rounds 1–3 are feasible under the WARP activity rule, yet the only feasible bundle in Round 4 is the zero bundle. This establishes:

Theorem 1. *The WARP activity rule satisfies Axioms 1 and 2, but fails to satisfy Axiom 3.*

Table 1: A "dead end" under the WARP activity rule

Round	**Clock Prices** (A, B, C)	**Demand** (A, B, C)
1	$p^1 \; = \; (1, 1, 1)$	$x^1 \; = \; (1, 0, 0)$
2	$p^2 \; = \; (1, 1, 4)$	$x^2 \; = \; (0, 1, 0)$
3	$p^3 \; = \; (4, 1, 4)$	$x^3 \; = \; (0, 0, 1)$
4	$p^4 \; = \; (4, 3, 5)$	

As was mentioned above, Axiom 2 implies Axiom 3 for straightforward bidding. Therefore, the bidding history in Table 1 must be irrational. It is easy to see that the choice of x^3 in round 3 violates rationality when taken together with the demand choices of rounds 1 and 2.[19] More generally, this example illustrates that the WARP activity rule exhibits an undesirable property that could be called *payoff discontinuity*. Suppose that the bidder whose bidding is displayed in Table 1 has a unit demand and values all three goods at 100, i.e., $v(A) \; = \; v(B) \; = \; v(C) \; = \; 100$. Given these values, the bidding shown in Table 1 is truthful in rounds 1 and 2, but contains

[19] In rounds 1 and 2, the bidder revealed that $v(x^3) \leq v(x^1) = v(x^2)$. In round 3, the bidder reveals that $v(x^2) \leq v(x^3) - 3$. Taken together, these inequalities are inconsistent with the existence of the value function that rationalizes demand choices in these rounds.

a mistake in round 3 (the bidder attains a payoff of 96 from its choice of good C, instead of the optimal payoff of 99 from choosing good B). Observe that the "small" mistake of round 3 triggers a massive payoff discontinuity in round 4; the activity rule requires the bidder to bid for the zero bundle (with an implied payoff of zero) in round 4, while bidding for any good would have resulted in a payoff of at least 95.

4.2 GARP Activity Rule

The definition of GARP for quasilinear preferences is as follows:

Definition 10 (GARP). *A collection of price-demand observations $(p^k,\, x^k)$, $k = 1, ..., t$, satisfies GARP for quasilinear preferences if for any choice of distinct indices $t_1, ..., t_s \in \{1, ..., t\}$,*

$$p^{t_1} \cdot [x^{t_1} - x^{t_s}] \;+\; p^{t_2} \cdot [x^{t_2} - x^{t_1}] \;+\; \dots \;+\; p^{t_s} \cdot [x^{t_s} - x^{t_{s-1}}] \;\leq\; 0. \tag{4.2}$$

GARP ensures that the bidder's demand choices cannot reveal a lack of rationality in the following sense. Suppose that a bidder initially owns bundle x^{t_s}. The demand choices made in rounds $t_1, ..., t_s$ imply that the bidder would be willing to participate in a series of trades that would eventually leave the bidder with the same bundle x^{t_s}. GARP requires that the corresponding total net payment that the bidder is willing to pay to be involved in these transactions (the left-hand side of (4.2)) should be nonpositive. A positive net payment implies the existence of a "money pump," which is inconsistent with rationality.

The activity rule based on GARP is defined as follows:

Definition 11 (GARP Activity Rule). *The feasible set of bids in Round $t \geq 2$ is $X^t = \{z \in \Omega :\; (p^1,\, x^1), ..., (p^{t-1},\, x^{t-1}), (p^t,\, z)$ satisfies quasilinear GARP$\}$.*

The main distinction between GARP and WARP is that GARP guarantees the existence of a value function that is consistent with the observed demands. This famous result is known as Afriat's Theorem (Afriat, 1967). The following version of Afriat's Theorem for quasilinear preferences is due to Rochet (1987).

Theorem 2. *[Rochet(1987)] The following statements are equivalent:*

(i) collection $(p^k,\, x^k)$, $k = 1, ..., t$, satisfies GARP for quasilinear preferences; and

(ii) There exists a value function that rationalizes collection $(p^k,\, x^k)$, $k = 1, ..., t$. In other words, there exists a set of numbers $\{v^k\}_{k=1}^t$ such that

$$v^i \;\leq\; v^j \;+\; p^j \cdot (x^i \;-\; x^j) \qquad \forall\, i, j \;=\; 1, \dots, t. \tag{4.3}$$

The existence of a value function that rationalizes the observed demand choices implies that the GARP activity rule satisfies all three axioms. This establishes:

Theorem 3. *The GARP activity rule satisfies Axioms 1, 2 and 3. Furthermore, the GARP activity rule is the strictest activity rule to satisfy these axioms.*

Proof. In the appendix.

A bidder might naively think that it is harmed by the extra strength of the GARP activity rule, since the rule is depriving the bidder of some of its available bids under the less stringent WARP activity rule. However, in reality, the GARP activity rule is saving the bidder from itself; the GARP rule stops the bidder from wandering inadvertently into a dead end. Our next theorem restates this property by showing that the GARP rule protects the bidder from experiencing a payoff discontinuity.

Theorem 4. *Under the GARP activity rule, suppose that a bidder bids straightforwardly according to $v(\cdot)$ in rounds $1, ..., t-1$, and bids for x^t in round t such that*

$$\pi(p^t, v) \; - \; [\, v(x^t) \; - \; p^t \cdot x^t \,] \; = \; \delta \; > \; 0 \,, \tag{4.4}$$

where $\pi(p, v) = \max_{z \in \Omega} \{ v(z) - p \cdot z \}$. Then there exists value function $\tilde{v}(\cdot) \in V$ that rationalizes the bidding history (p^k, x^k), $k = 1, ..., t$ and

$$|\, \tilde{v}(z) \; - \; v(z) \,| \; \leq \; \delta \qquad \forall \; z \; \in \; \Omega \,. \tag{4.5}$$

Proof. In the appendix.

Intuitively, Theorem 4 establishes that a straightforward bidder who makes a mistake that forfeits payoff of δ in a given round can always continue to bid in all future rounds according to a modified value function that differs from its true value function by at most δ. As a result, the cost to the bidder of this mistake never exceeds δ in any future round. Unlike the WARP activity rule, which was shown in the last section to suffer from payoff discontinuity, the GARP activity rule benefits from payoff continuity.

Our next result establishes that validating GARP is the only way to simultaneously satisfy all three axioms. This result is proved for an auction environment with nonlinear bundle prices where the auctioneer quotes a price vector $p^t \in \mathbb{R}_+^{|\Omega|}$ (i.e., a separate price for each bundle) in each round t. To avoid any ambiguity, let us note that Theorem 5 is the only result in this paper that allows nonlinear bundle prices.[20]

Theorem 5. *When nonlinear bundle prices are allowed, the GARP activity rule is the unique activity rule that satisfies Axioms 1, 2 and 3.*

Proof. In the appendix.

[20] The corresponding definitions of WARP and GARP for nonlinear bundle prices are provided within the proof of Theorem 5 in the appendix.

The proof of Theorem 5 uses the following logic. When a bidder commits a GARP violation in Round t, the auctioneer gains the ability to select a "checkmate" price vector p^{t+1} such that all prior demands violate Axiom 1 at these prices, in turn leading to the failure of Axiom 3. When constrained, this price vector can be selected such that $p^{t+1} \geq p^t$ to form a non-decreasing price path. Nonlinear bundle prices always provide enough degrees of freedom to construct such a price vector. The same argument applies to environments with linear prices and a bidder with unit demand (like the example in Table 1). With linear clock prices but multi-unit demand, the number of degrees of freedom might be insufficient to construct a checkmate price vector p^{t+1}, but the auctioneer can still set prices such that many prior demands become infeasible (including the demand from the last round) potentially causing a major payoff discontinuity.

Unlike point monotonicity, the GARP activity rule might disallow bidding for the bundle demanded in the previous round. From a practical perspective, it is important for bidders to understand which bundles will be feasible in future rounds. By Axiom 3, one of the previously demanded bundles is always feasible. Proposition 2 provides a further characterization of the set of feasible bids when prices are non-decreasing.

Proposition 2. *For non-decreasing price paths, the following statements are true for the GARP activity rule:*

(a) ***(feasibility of smaller bids)*** *If* $z \in X^t$, *then* $z' \in X^t$ *for all* $z' \leq z$.

(b) ***(feasibility of non-cost-increasing items)*** *Bundle* z *defined by*

$$z(m) = \begin{cases} x^{t-1}(m), & \text{if} \quad p^t(m) = p^{t-1}(m) \\ 0, & \text{if} \quad p^t(m) > p^{t-1}(m) \end{cases} \qquad \forall m = 1, ..., M$$

is feasible in Round $t \geq 2$.

Proof. In the appendix.

4.3 SARP Activity Rule

The GARP activity rule has been shown to be essentially the unique activity rule that satisfies Axioms 1, 2 and 3. However, Axiom 2 can be viewed as being too permissive in the sense that all potential profit-maximizing bundles must be feasible. Axiom $2'$ is a variant on Axiom 2 that allows the auctioneer to narrow the set of feasible bids in certain situations.

To illustrate the difference, consider an example with two unique goods, A and B, and a bidding history provided in Table 2. Between Rounds 1 and 2, the price of good A did not change and the price of good B went up. As a result, the cost of bundles $x^1 = (1, 1)$ and $x^2 = (0, 1)$ have increased by the same amount, and the shift in demand from x^1 to x^2 reveals that the bidder has been indifferent between these bundles in both rounds. If an activity rule were to disallow bidding for bundle x^2 in round 2, it would have violated Axiom 2, but it would have satisfied Axiom $2'$ since the bidder can still bid for x^1.

Table 2: A bidding history with a revealed indifference.

Round	**Clock Prices** (A, B)	**Demand** (A, B)
1	$p^1 = (1, 1)$	$x^1 = (1, 1)$
2	$p^2 = (1, 2)$	$x^2 = (0, 1)$

Observe that the bidder in this example has demonstrated an inconsistent tie-breaking by first preferring x^1 to x^2 in Round 1, and then preferring x^2 to x^1 in Round 2. The auctioneer can require a consistent tie-breaking among potential indifferences: if a bidder was indifferent between bundles z and z' in round t and has chosen bundle z, then the bidder must continue to break future direct (and indirect) ties between these bundles in favor of bundle z. Let $r(\cdot)$ denote a ranking order, a function that maps Ω into set $\{1, 2, ..., |\Omega|\}$ with a property that $r(z) \neq r(z')$ for any pair of bundles $z \neq z'$.

Definition 12 (Consistent Tie-Breaking)**.** *A bidder is said to bid straightforwardly according to value function $v(\cdot)$ and exhibit consistent tie-breaking if there exists a ranking order $r(\cdot)$ such that the bidder's demand vector x^t has the lowest rank among all bundles in its current demand correspondence $D(p^t, v)$ in each round t, i.e.,*

$$x^t = \operatorname*{arg\,min}_{z \in D(p^t, v)} r(z) \qquad \forall t. \tag{4.6}$$

The rationality concept that enforces consistent tie-breaking (i.e., disallows revealing indifferences) is known as *the strong axiom of revealed preference* (SARP).[21]

Definition 13 (SARP)**.** *A collection of price-demand observations (p^k, x^k), $k = 1, ..., t$, satisfies SARP for quasilinear preferences if for any choice of distinct indices $t_1, ..., t_s \in \{1, ..., t\}$ such that $x^{t_1} \neq x^{t_s}$,*

$$p^{t_1} \cdot [x^{t_1} - x^{t_s}] + p^{t_2} \cdot [x^{t_2} - x^{t_1}] + \ldots + p^{t_s} \cdot [x^{t_s} - x^{t_{s-1}}] < 0. \tag{4.7}$$

Theorem 6 provides the analog of Afriat's Theorem for SARP.

Theorem 6. *The following statements are equivalent:*

(i) collection (p^k, x^k), $k = 1, ..., t$, satisfies SARP for quasilinear preferences; and

[21] SARP is generally known as a rationality concept for strict preferences and domains with fully divisible goods. With indivisible goods, multiple-valued demands are unavoidable.

(ii) *There exists a value function that rationalizes collection* (p^k, x^k), $k = 1, ..., t$, *and exhibits consistent tie-breaking. In other words, there exists a set of numbers* $\{v^k\}_{k=1}^t$ *such that*

$$v^i \le v^j + p^j \cdot (x^i - x^j), \qquad \forall\, i, j \in \{1, \dots, t\}, \tag{4.8}$$

and a ranking order $r(\cdot)$ *such that*

$$r(x^j) \le r(x^i) \qquad \forall\, i, j : v^i = v^j + p^j \cdot (x^i - x^j). \tag{4.9}$$

Proof. In the appendix.

The corresponding SARP activity rule is defined as follows:

Definition 14 (SARP Activity Rule)**.** *The feasible set of bids in Round* $t \ge 2$ *is* $X^t = \{z \in \Omega : (p^1, x^1), ..., (p^{t-1}, x^{t-1}), (p^t, z)$ *satisfies quasilinear SARP*$\}$.

The example in Table 2 violates SARP since $p^1 \cdot [x^1 - x^2] + p^2 \cdot [x^2 - x^1] = 0$. However, the bidder can continue to bid on any of (1, 1), (1, 0) or (0, 0) under this activity rule. Note that such a limitation on bidders' demands can be quite desirable in practice. Scenarios where bidders reduce their demands for goods that have not increased in price can be very disruptive for an auction's performance. Furthermore, the ability to reduce demand at will unlocks many strategic bidding opportunities for a bidder since it never bears a significant risk of "getting stuck." The SARP activity rule naturally creates such risk for strategic bidders and does it in a non-harmful way for straightforward bidders. At the same time, the fabled Walrasian auctioneer could find Axiom $2'$ and the SARP activity rule to be less than satisfactory, as reference to the entire demand correspondence may be needed to reach a Walrasian equilibrium.

With nonlinear bundle prices, the GARP activity rule is the weakest activity rule that satisfies Axioms 1, $2'$ and 3. Theorem 7 proves that the SARP activity rule is the strictest activity rule that satisfies these axioms.

Theorem 7. *The SARP activity rule is the strictest activity rule that satisfies Axioms 1,* $2'$ *and 3.*

Proof. In the appendix.

The SARP activity rule should be viewed as one possible refinement of the GARP activity rule. Other refinements are also possible. For example, GARP allows straightforward bidding according to any quasilinear value function. The auctioneer can strengthen the activity rule by incorporating additional restrictions on the valuation domain that can be derived from specifics of a given auction setting. For example, the auctioneer can restrict bidders to value functions that are additively separable among subsets of goods.

5 Learning and Value Updating

One of the important rationales for using dynamic auctions is to provide bidders with opportunities for updating their value estimates in response to their rivals' bidding during the auction (i.e., learning). This has been argued to be especially beneficial in environments with interdependent values or allocative externalities.

It is important to emphasize at the outset that accommodating value updates is to a certain extent contrary to the main premise for activity rules. An activity rule that allows bidding according to a rapidly-changing value function will have to sacrifice some of its power in maintaining an orderly bidding process. For example, a large update to a value function can trigger a violation of the Law of Demand, requiring a relaxation of Axiom 1. But under a weakened version of Axiom 1, bidders' opportunities to engage in delayed bidding and parking would become greater.

We consider the question of accommodating value function updates in two steps. First, in this section, we investigate the extent to which the pure GARP activity rule—without any modification—allows value updates (see Theorem 8, below). Then, in the next section, we introduce possible relaxations of the GARP activity rule for situations in which the auctioneer believes that the pure GARP activity rule is too restrictive and would preclude bidders from making legitimate updates.

Suppose that a bidder's value function can change from one round to the next, and let $v^t(\cdot)$ denote its value function in round t. (For example, $v^t(\cdot)$ can be interpreted as the bidder's estimate of its true value function $v(\cdot)$ in round t.) In addition, let the marginal value of bundle z relative to bundle z' in round t be denoted by

$$mv^t(z, z') \;=\; v^t(z) \;-\; v^t(z'). \tag{5.1}$$

The GARP condition (4.2) for a sequence $t_1, ..., t_s \in \{1, ..., t\}$ can be restated in an alternative form:

$$\sum_{k=1}^{s-1} (p^{t_{k+1}} \;-\; p^{t_k}) \cdot (x^{t_s} \;-\; x^{t_k}) \;\leq\; 0. \tag{5.2}$$

The alternative formulation (5.2) states that bundle x^{t_s} cannot be too expensive when demanded in round t_s (compared to the bundles demanded in rounds $t_1, t_2, ..., t_{s-1}$) in order to be consistent with GARP. Proposition 3 below establishes a link between value function updates and relative price changes (expressed in the same form as (5.2)) for a straightforward bidder who is not constrained by an activity rule.

Proposition 3. *If a bidder bids straightforwardly according to value functions $v^1(\cdot)$, ..., $v^t(\cdot)$ in rounds $1, ..., t$, respectively, then for any choice of distinct indices $t_1, ..., t_s \in \{1, ..., t\}$,*

$$\sum_{k=1}^{s-1} (p^{t_{k+1}} - p^{t_k}) \cdot (x^{t_s} - x^{t_k}) \;\;\leq\;\; \sum_{k=1}^{s-1} \big[mv^{t_{k+1}}(x^{t_s}, x^{t_k}) \;-\; mv^{t_k}(x^{t_s}, x^{t_k})\big]. \tag{5.3}$$

Observe that condition (5.3) is equivalent to GARP formulations (4.2) and (5.2) when the value function does not vary from round to round (the right-hand side of (5.3) is zero when $v^t(\cdot) = v(\cdot)$ for all t). For a round-dependent value function, condition (5.3) states that the cumulative marginal value update for bundle x^{t_s} relative to bundles $x^{t_1}, ..., x^{t_{s-1}}$ always exceeds the cumulative price change for bundle x^{t_s} relative to the same bundles when a bidder bids straightforwardly.

The implications of this condition are best illustrated for the special case of a two-observation subsequence $t_1, t_2 \in \{1, ..., t\}$, where $t_1 \leq t_2$:

$$(5.4) \qquad (p^{t_2} - p^{t_1}) \cdot (x^{t_2} - x^{t_1}) \leq mv^{t_2}(x^{t_2}, x^{t_1}) - mv^{t_1}(x^{t_2}, x^{t_1}).$$

Straightforward bidding in rounds t_1 and t_2 implies that the update of the marginal value $mv(x^{t_2}, x^{t_1})$ that occurred between these rounds (the right-hand side of (5.4)) weakly exceeds the corresponding price change for these bundles (the left-hand side of (5.4)). When the price change is positive and bidding for x^{t_2} in round t_2 would violate revealed preference, the marginal value update that occurred between rounds t_1 and t_2 would have to be rather large—the minimum update must weakly exceed the price change! In other words, if bundle x^{t_2} has increased in price relative to bundle x^{t_1} by 100 between rounds t_1 and t_2, the value update for x^{t_2} relative to x^{t_1} must be at least 100 to justify this violation under straightforward bidding.

More generally, suppose that a bidder bids straightforwardly according to value functions $v^1(\cdot), ..., v^{t-1}(\cdot)$ in rounds $1, ..., t-1$, respectively, and that the bidding history (p^1, x^1),...,(p^{t-1}, x^{t-1}) satisfies GARP. When would bidding for bundle $z \in D(p^t, v^t(\cdot))$ in round t be consistent with the GARP activity rule? Motivated by condition (5.3), our sufficient condition puts an upper limit on possible value function updates to ensure that any bundle infeasible under GARP cannot be demanded by a straightforward bidder.

Definition 15 (Price-Justified Updating)**.** *Given round prices $p^1, ..., p^t$, value functions $v^1(\cdot), v^2(\cdot), ..., v^t(\cdot)$ satisfy the "price-justified updating" property if, for any choice of corresponding demands $x^1, ..., x^t$ (i.e., $x^k \in D(p^k, v^k(\cdot))$ for all $k = 1, ..., t$) and for any choice of distinct indices $t_1, ..., t_s \in \{1, ..., t\}$ such that*

$$(5.5) \qquad \sum_{k=1}^{s-1} (p^{t_{k+1}} - p^{t_k}) \cdot (x^{t_s} - x^{t_k}) > 0,$$

the cumulative marginal value update for bundle x^{t_s} relative to bundles $x^{t_1}, ..., x^{t_{s-1}}$ that occurred in rounds $t_2, ..., t_s$ is strictly less than the cumulative price change for bundle x^{t_s} relative to the same bundles in the same rounds, i.e.:

$$(5.6) \qquad \sum_{k=1}^{s-1} \left[mv^{t_{k+1}}(x^{t_s}, x^{t_k}) - mv^{t_k}(x^{t_s}, x^{t_k})\right] < \sum_{k=1}^{s-1} (p^{t_{k+1}} - p^{t_k}) \cdot (x^{t_s} - x^{t_k}).$$

Condition (5.6) is intuitive: if bundle x^{t_s} is infeasible under the GARP activity rule, then its value update relative to bundles demanded in prior rounds must be relatively small. Imposing an upper limit on updates ensures that even with an updated value, bidding for bundle x^{t_s} still contradicts straightforward bidding.

Theorem 8. *The GARP activity rule allows straightforward bidding according to value functions $v^1(\cdot)$, ..., $v^t(\cdot)$ in rounds $1, ..., t$ with prices $p^1, ..., p^t$, respectively, if these functions satisfy the "price-justified updating" property.*

Proof. In the appendix.

Updates in excess of relative price changes can be viewed as unjustified in many practical settings. For example, consider a bidder who faces common value uncertainty characterized by a single peaked distribution. In such an environment, learning is limited to observing clock prices (and changes in opponents' demands if reported), and the value updates should not exceed the relative price changes. Another important class of updates permitted by the GARP activity rule is updates to absolute values (values relative to the zero bundle): for each round t and given a round-specific constant α^t, $v^t(z) = v(z) + \alpha^t$ for all $z \neq \vec{0}$. Such updates trivially satisfy (5.6) so long as the prices are non-decreasing and the bidder never bid for the zero bundle in any prior round.

In practice, dynamic auctions present bidders with multiple channels of information that can be used for updating, and the updates do not have to be proportional to observed changes in clock prices, leading to violations of "price-justified updating". A good example is a setting with allocative externalities when the value of the winning bundle for a bidder depends on the allocation of the other goods among her opponents. While the GARP activity rule might get in the way of such updates, it is an open question whether some of these updates should be recognized by the auctioneer since frequently such private values run contrary to social value.[22]

Finally, it is instructive to compare the GARP activity rule with the commonly-used point-monotonic activity rule from the perspective of facilitating updates. In general, an activity rule that accommodates straightforward bidding according to a rapidly-fluctuating value function would violate Axiom 1 and satisfy Axiom 2. Observe that the point-monotonic rule fails both axioms, while the GARP activity rule satisfies both axioms. As a result, the ranking of these rules is ambiguous; consequently, it is easy to construct examples where point monotonicity is more restrictive in relation to updating than the GARP activity rule, and vice versa.

[22] For example, incumbent wireless operators in a spectrum auction might attach a positive foreclosure value to denying licenses to new entrants, but regulators would presumably not want to recognize the incumbent bidders' preferences.

6 Relaxations of the GARP Activity Rule

In applications, the GARP activity rule might be too constraining for bidders. For example, the highly-utilized SMRA format routinely limits bidders' demand choices by assigning provisional winnings after each round. Other practical considerations that can overconstrain bidders under this rule are budget constraints, bidding mistakes and value updates that go beyond the "price-justified updating" property of Section 5. In this section, we describe several ways in which the GARP activity rule can be relaxed to accommodate these factors.

Incorporating Provisional Winnings

The origin of many modern auction designs can be traced to the SMRA format. These auctions use a notion of "provisional winnings"—after each round the auctioneer determines a tentative winning allocation that will prevail if no new bids are made in the next round. The bidder is held to its provisional winnings even if the provisional winnings are no longer part of the bidder's demand. As such—and in contrast to our modeling—a bidder with provisional winnings is constrained in choosing its demand report and this constraint might prevent the bidder from bidding truthfully.

Provisional winnings create two challenges for the GARP activity rule. First, a bidder might face nonlinear bundle prices $p^t \in \mathbb{R}_+^{|\Omega|}$ that depend on current clock prices for new bids, but on earlier clock prices for provisional winnings. This complication is technical and can be handled easily by restating the revealed preference constraints in terms of nonlinear prices. Second, a bidder with provisionally winnings might be forced to bid untruthfully and, in that event, the bidder's potentially-suboptimal choice should not be held against the bidder in future rounds. To put it differently, a bidder should be permitted to resume truthful bidding when it is no longer constrained by provisional winnings. This complication is conceptual and it requires a change to the GARP formulation.

To accommodate provisional winnings, we additionally assume that the true value functions are monotonic.

(A3) Monotonicity: A bidder's true value function $v(\cdot)$ is monotonic, i.e.,
$v(z') \leq v(z)$ for any bundles $z, z' \in \Omega$ such that $z' \leq z$.

For a given bidder, let $w^k \in \Omega$ denote a provisionally-winning bundle in round k. Suppose that the bidder is limited to choosing its demand report x^k in round k from set $X^k \cap \Omega^k$ where $\Omega^k = \{ z \in \Omega : z \geq w^k \}$. Under this constraint, the bidder's choice in round k might be suboptimal. To ensure that the bidder is able to resume straightforward bidding in future rounds, the auctioneer must relax revealed preference constraints generated in relation to the bid of round k.

For any bundle $z' \in \Omega$, define bundle $z = z' \vee w^k$ to be the component-wise maximum of bundles z' and w^k. Since $z \in \Omega^k$ and $z' \leq z$, the revealed preference constraint in round k

between bundles z' and x^k can be relaxed as follows:

$$v(z') \;\le\; v(z) \;\le\; v(x^k) \;+\; p^k(z) \;-\; p^k(x^k). \tag{6.1}$$

A bidding history with provisional winnings can be tested for GARP using the following definition.

Definition 16 (GARP with Provisional Winnings)**.** *A collection of price-demand-provisional winnings observations* $(p^k,\, x^k,\, w^k)$, $k = 1, ..., t$, *satisfies GARP with provisional winnings for preferences satisfying (A1)–(A3) if there exists a set of numbers* $\{v^k\}_{k=1}^t$ *such that*

$$v^i \;\le\; v^j \;+\; p^j(x^i \vee w^j) \;-\; p^j(x^j), \qquad \forall\, i, j \;=\; 1, \dots, t. \tag{6.2}$$

Incorporating Budget Constraints

In many applications, the presence of budget constraints is a real concern, and bidders might violate (quasilinear) GARP even when they bid truthfully subject to a budget. To accommodate budget-constrained bidders, the auctioneer can adopt the following relaxation of GARP (taken from Harsha et al. (2010)).

Definition 17 (GARP with Budget)**.** *A collection of price-demand observations* $(p^k,\, x^k)$, $k = 1, ..., t$, *satisfies GARP with a budget for preferences satisfying (A1)–(A2) if there exists a set of numbers* $\{v^k\}_{k=1}^t$ *and a budget* $B \ge 0$ *such that*

$$\begin{array}{lll} (1) & v^i \;\le\; v^j \;+\; p^j \cdot (x^i \;-\; x^j), & \forall\, i, j \,\in\, \{1, ..., t\} :\; p^j \cdot x^i \;\le\; B; \\ (2) & p^j \cdot x^j \;\le\; B, & \forall j \,\in\, \{1, ..., t\}. \end{array} \tag{6.3}$$

This relaxation allows bidders to bid straightforwardly, staying within their budgets (i.e., bidding on the most profitable affordable bundle). Also observe that it is relatively costly for a strategic bidder to abuse this rule—a GARP violation reveals the corresponding budget and, at higher clock prices, the budget constraint may squeeze the bidder out of the auction.

Allowing Bounded Rationality

To allow extra room for bidding errors and non-justified value updates, the auctioneer can permit some amount of irrationality in the bidding history. For example, the auctioneer can admit cycles that do not exceed a predefined limit: this is easily accomplished by replacing the right-hand side of the GARP formula (4.2) with a positive number. A more direct approach is to manually add relaxation parameters into the system of revealed preference constraints.

Definition 18 (Relaxed GARP)**.** *Given relaxation parameters* $\lambda^k \ge 0$, $k = 1, ..., t$, *a collection of price-demand observations* $(p^k,\, x^k)$, $k = 1, ..., t$, *satisfies relaxed GARP for preferences satisfying (A1)–(A2) if there exists a set of numbers* $\{v^k\}_{k=1}^t$ *such that*

$$\begin{array}{lll} (1) & v^i \;\le\; v^j \;+\; p^j \cdot (x^i \;-\; x^j) \;+\; \lambda^j, & \forall\, i, j \,\in \{1, ..., t\}; \\ (2) & v^i \;=\; v^j, & \forall\, i, j :\;\; x^i \;=\; x^j. \end{array} \tag{6.4}$$

A relaxation parameter $\lambda^k \geq 0$ can be interpreted as a discount applied in round k to the price of bundle x^k. There are numerous ways to set the relaxation parameters, and their exact choice should be tailored to the specifics of a particular auction. For example, setting $\lambda^k = \epsilon\, p^k \cdot x^k$ corresponds to applying a discount of ϵ (expressed as a percentage) to the demanded bundle. In case a dependence on x^k is undesirable, the auctioneer can set $\lambda^k = \epsilon\, p^k \cdot z$ (where z is some predefined bundle), allowing a bidder to make mistakes in an $\epsilon\, z$-neighborhood of its true demand without overconstraining itself in future rounds.

Hybrid Revealed Preference Activity Rules

Point monotonicity is the most widely used activity rule in spectrum auctions, yet as we have seen, it may preclude straightforward bidding. This problem was well understood in some recent spectrum auctions by regulators, who adopted rules allowing an exception based on a specific set of WARP inequalities. In the next section, we will demonstrate several strategic exploits that were available to bidders via this exception.

To close this loophole, ISED, Canada's spectrum regulator, adopted a new activity rule for its 600 MHz spectrum auction held in 2019. The logic behind the new rule is the same: follow point monotonicity, but allow exceptions when a bidder has to violate point monotonicity in order to bid straightforwardly. However, instead of testing a rather limited subset of WARP inequalities, the new rule tests the relevant subset of the bidder's bidding history for full GARP compliance.

Formally, let E^t denote the minimum number of eligibility points associated with any demand bid in rounds 1 through $t-1$, i.e.,

$$E^t = \min_{s \in \{1,\ldots,t-1\}} E(x^s), \qquad \text{for all} \quad t \geq 2. \tag{6.5}$$

A hybrid of a GARP activity rule and a point-monotonic activity rule can be defined for non-decreasing prices as follows:

Definition 19 (Hybrid GARP/Point Monotonicity)**.** *The feasible set of bids in Round $t \geq 2$ is $X^t = \{z \in \Omega :\ E(z) \leq E^t$ or $(p^s, x^s), (p^{s+1}, x^{s+1}), \ldots, (p^{t-1}, x^{t-1}), (p^t, z)$ satisfies quasilinear GARP, where s is the last round in which $E^s \geq E(z)\,\}$.*

This activity rule trivially satisfies Axioms 2 and 3, but it fails Axiom 1 since it is based partly on point monotonicity.

7 Evidence from Spectrum Auctions

Is the GARP activity rule practical for use in the real world? To a very limited extent, one can investigate this question by taking actual bidding data from spectrum auctions and testing for GARP consistency. When bids pass the GARP test, the answer is unambiguously affirmative. However, when bids fail the GARP test, there are two rather different possible interpretations. On the one hand, a failure of GARP consistency might suggest that the activity rule is too onerous. On the other hand, certain bids may fail the GARP test because current activity rules are too weak—and the spectrum regulator may prefer if the activity rule would prevent such bids.

Under the first interpretation, the GARP check is a test for detecting situations where, for example, the bidders' make substantial updates to their relative values and bidding according to consistent values is unrealistic. Under the second interpretation, the GARP check is an effective test for detecting instances of strategic manipulation or opportunistic behavior. These two interpretations seem to be largely empirically indistinguishable—and the current paper cannot hope to obtain a definitive answer as to the better interpretation. All that we do in this section is to initiate a discussion.

Our choice of spectrum auctions for this empirical exercise was constrained by two requirements: (1) the auction rules established no provisionally-winning licenses (or established them in limited numbers); and (2) the full bidding data was released into the public domain.[23] This left us with bidding data from just four recent auctions: three that used the CCA format and one that used an SMRA-like format:[24,25]

- ***UK 4G Auction (2013)*** This auction used the CCA format to allocate nationwide spectrum licenses grouped as 6 product categories (A1, A2, C, D1, D2 and E), with supplies ranging from 1 block in the A2 category to 14 blocks in the C category. Seven bidders participated. The clock stage of the auction used a point-monotonic activity rule and ran for 52 rounds. The auction allocated all available licenses and generated £2.34 billion in revenues.

- ***Canadian 700 MHz Auction (2014)*** This auction used the CCA format to allocate spectrum licenses in 14 regions, each including 7 licenses grouped as 4 product categories (A, B/C, C1/C2 and D/E). Ten bidders participated. The clock stage of the auction

[23]Note that the first criterion prevented use of US spectrum auction data, while the second criterion prevented use of most other countries' spectrum auction data.

[24]Auction rules, results and detailed bidding data for each auction can be found at: [UK 4G] `https://www.ofcom.org.uk/spectrum/spectrum-management/spectrum-awards/awards-archive/800mhz-2.6ghz`; [Canada 700 MHz] `https://www.ic.gc.ca/eic/site/smt-gst.nsf/eng/h_sf01714.html`; [Canada 2500 MHz] `https://www.ic.gc.ca/eic/site/smt-gst.nsf/eng/h_sf10939.html`; [UK 2.3 and 3.4 GHz] `https://www.ofcom.org.uk/spectrum/spectrum-management/spectrum-awards/awards-archive/2-3-and-3-4-ghz-auction`

[25]The authors disclose that: (i) they advised ISED, the Canadian spectrum regulator, on the design and implementation of the 700 MHz and 2500 MHz auctions; and (ii) they advised Three UK (H3G) on bidding strategy in the UK 4G and UK 2.3/3.4 GHz auctions. To avoid potentially biasing the analysis of this section, the authors refrain from discussing H3G's bidding in these auctions, except for reporting summary statistics for H3G's GARP violations in Table 3.

used a hybrid WARP/point-monotonic activity rule and ran for 106 rounds. The auction allocated 97 (out of 98) licenses and generated \$5.27 billion CAD (Canadian dollars) in revenues.

- ***Canadian 2500 MHz Auction (2015)*** This auction used the CCA format to allocate spectrum licenses in 61 regions, each including varying number of licenses grouped as 2 product categories. Eleven bidders participated. The clock stage of the auction used a hybrid WARP/point-monotonic activity rule and ran for 39 rounds. The auction allocated 302 (out of 318) licenses and generated \$755 million CAD in revenues.

- ***UK 2.3 and 3.4 GHz Auction (2018)*** This auction used an SMRA-like format to allocate four nationwide licenses in the 2.3 GHz band and 30 nationwide licenses in the 3.4 GHz band.[26] Five bidders participated. The auction used a point-monotonic activity rule and ran for 67 rounds. The auction allocated all available licenses and generated £1.37 billion in revenues.

For each auction-bidder pair, we assess the GARP consistency of the bidding history by minimizing the sum of relaxation terms $\{\lambda^k\}_{k=1}^T$ from the relaxed GARP formulation (6.4).[27] When the minimized sum is zero, the bidding history is fully compliant with GARP. A positive sum indicates that the bidding history is not consistent with GARP and its amount corresponds to the total amount of relaxation that is needed to rationalize the bidding history. We report results of minimization in Table 3 in the "Total Violation" column. To provide a measure of magnitude for GARP violations relative to the bidder's size, we report the average violation and the maximum violation across all rounds as a percentage of the bid amount, evaluated at current prices (i.e., the average and maximum ϵ for $\lambda^k = \epsilon\, p^k\, x^k$ in each round k).[28]

We also investigate the sources of GARP violations. Auctions with activity rules that use eligibility points are especially vulnerable to manipulation when a bidder switches among different packages without reducing its eligibility. Unless the auction is near its end, such switches in demand have almost no consequences for a bidder and, as a result, the bidder can mostly disregard relative price movements. To assess the extent to which GARP violations can be attributed to these rounds, we partition each bidding history into subsets corresponding

[26] The auction format was an amalgam of the SMRA and the clock auction formats. As in an SMRA, this auction design included provisional winnings, but as in a clock auction, bidders bid for quantities of blocks in each band rather than for individual licenses. The auction rules also allowed each bidder to use up to three eligibility waivers to avoid its eligibility being reduced due to low activity. For our analysis, we excluded those rounds in which a given bidder took a waiver.

[27] For the UK 2.3 and 3.4 GHz auction that used an SMRA-like format, we apply: (1) a combination of relaxed GARP formulations (6.2) and (6.4) to account for provisional winnings; and (2) the round's prevailing clock prices for pricing all packages, including provisional winnings.

[28] When calculating the maximum violation, we minimize the maximum relaxation, $\max \lambda^k$, over all rounds instead of minimizing the sum of $\sum \lambda^k$.

to different levels of eligibility and we check each subset individually. Lastly, we calculate the proportion of GARP violations that can be attributed to violations of the Law of Demand. We do this by minimizing the sum of relaxation terms, $\{\lambda^k\}_{k=1}^{T}$, for a relaxed WARP formulation analogous to (6.4). The results of these two exercises are reported in the last two columns of Table 3 as percentages of the total GARP violation.

Table 3: Analysis of GARP violations in recent spectrum auctions

Bidders	**Absolute and Relative GARP Violation**			**Source of Violation**	
	Total Violation (millions)	Average % Violation	Maximum % Violation	In Eligibility Preserving Rounds	From Law of Demand Violations
UK 4G Auction (2013, CCA, 52 rounds, 7 bidders)					
EE	£591.54	2.0%	18.9%	47.2%	98.5%
Vodafone	£159	0.3%	5.4%	100%	100%
H3G	£30.14	0.1%	1.2%	99.5%	99.8%
Niche	£2.25	0.02%	0.5%	0%	100%
Telefonica, MLL, HKT	*no GARP violations*				
Canadian 700 MHz Auction (2014, CCA, 106 rounds, 10 bidders)					
TELUS	$2649.6	1.7%	7.1%	20.4%	67.1%
Videotron	$1889.0	4.4%	27.8%	48.2%	93.9%
Bell	$1415.8	1.3%	12.9%	59.1%	71.4%
Rogers	$636.0	0.3%	1.6%	0%	53.6%
Bragg	$254.7	5.3%	27.8%	39.0%	80.7%
SaskTel	$188.3	7.4%	47.6%	100%	100%
MTS	$60.6	1.6%	13.5%	1.8%	81.6%
Feenix	$37.6	1.8%	11.0%	83.0%	83.4%
Novus	$8.7	0.3%	5.1%	100%	100%
TbayTel	$2.9	1.5%	14.2%	100%	100%
Canadian 2500 MHz Auction (2015, CCA, 39 rounds, 11 bidders)					
Bell	$87.5	4.1%	12.9%	100%	89.0%
Bragg	$53.9	7.4%	19.7%	89.3%	85.6%
Xplornet	$12.1	0.6%	2.9%	52.4%	72.4%
TELUS	$10.1	0.04%	0.4%	100%	100%
TbayTel	$2.1	5.0%	55.9%	100%	74.1%
CCI	$0.98	0.5%	2.5%	14.9%	64.6%
Rogers, Videotron, MTS, SSI, WIND	*no or minor GARP violations*				
UK 2.3 and 3.4 GHz Auction (2018, SMRA, 67 Rounds, 5 bidders)					
Vodafone	£27.85	0.2%	4.0%	24.0%	40.1%
Airspan, H3G, Telefonica, EE	*no GARP violations*				

While the results in Table 3 cannot be interpreted definitively, we are able to make several observations that suggest the general weakness of point-monotonic activity rules. First, a number of bidders bid fully consistently with GARP. Second, for those bidders who violated GARP, the average GARP violation is generally small. Third, a substantial proportion of GARP violations occurred in eligibility-preserving rounds and included violations of the Law of Demand.

We illustrate these points with detailed discussions of a few instances of bidding inconsistent with GARP, corresponding to the largest violations from Table 3.

Eligibility-Preserving Rounds: SaskTel in Canada's 700 MHz Auction

This example highlights the weakness of point-based activity rules in scenarios where a bidder switches its demand without reducing its eligibility. We examine SaskTel, which exhibits among the largest GARP violations in Table 3. 100% of SaskTel's violations occurred in such eligibility-preserving rounds. From round 55 through the end of the clock stage, SaskTel switched *five* times between demanding a B/C block and a $C1/C2$ block (each requiring 20 eligibility points) in the Saskatchewan region. Bids placed in these rounds accounted for 93% of Sasktel's total GARP violation.

Table 4: Switches between B/C and $C1/C2$ blocks by SaskTel

Round	**Bid**	**Eligibility Points**	**Revealed Value Difference** $v(B/C) - v(C1/C2)$
55 – 56	B/C	20	$\geq$ \$6.1 mil
57 – 68	$C1/C2$	20	$\leq$ \$2.5 mil
69 – 70	B/C	20	$\geq$ \$3.3 mil
71 – 72	$C1/C2$	20	$\leq$ \$4.0 mil
73 – 91*	B/C	20	$\geq$ \$26.5 mil
92 – 106	$C1/C2$	20	$\leq$ \$28.5 mil

* – *excluding round 88*

Table 4 reports the revealed value differential between B/C and $C1/C2$ blocks. Given prevailing prices, SaskTel's bids in rounds 55–56 implied a premium of at least \$6.1 million for the B/C block. However, its bids in rounds 57–68 implied a premium of less than \$2.5 million—an obvious inconsistency. SaskTel went on to reveal a premium of as much as \$26.5 million in later clock rounds, and finally reported a \$5 million premium in the supplementary round. Such bidding data is hardly consistent with any form of rational valuation, but it is consistent with bidding manipulation intended to drive up the relative price of B/C blocks (while SaskTel eventually won a $C1/C2$ block). And this manipulation was enabled by a fairly weak activity rule that did not preclude such demand movements.

Eligibility-Reducing Rounds: Bragg in Canada's 700 MHz Auction

Eligibility-reducing rounds have taken on a special role in most CCA auctions conducted to date, given that the revealed preference constraints have been applied selectively against the rounds when the bidder bids less than its eligibility.[29] As a result, a bidder who is about to reduce its eligibility has a strong incentive to choose a package that is likely to become more expensive in future rounds (instead of bidding truthfully). Such strategic bidding earns the bidder higher bid limits for its supplementary bids.

The most conspicuous example of such behavior was exhibited by Bragg in the Canadian 700 MHz auction (see Table 5). In Round 37, Bragg reduced its eligibility by 16 points. Its eligibility-reducing bid looks very odd since Bragg never expressed any interest in buying any licenses in Northern Quebec (NQC) or Alberta (AB) in any other rounds. Note that Bragg reverted to its original regions in Round 38 with a bid of the same size, so Round 38 is not an eligibility-reducing round. To sum up, Bragg apparently dropped the B/C block in Northern Ontario in two steps rather than one to obtain a strategic benefit. A similar move occurred in Round 68, when Bragg further reduced its eligibility by 31 points.

Table 5: Strategic Bidding by Bragg

Round	**Bid**								**Eligibility**
	NL	NS	NB	NQC	NON	MB	SK	AB	**Points**
Rounds 36–38									
36	C1+D	C1+D	C1+D		B+C1+D				109
37				A				A	93
38	C1+D	C1+D	C1+D		C1+D				93
Rounds 67–69									
67	B+D	B+D	B+D		B+D				93
68	A	A	A		A				62
69	B	B	B		B				62
Rounds 78–95									
78 – 93	C1	C1	C1		C1				62
94	C1	C1	C1		C1	B	B		106
95	C1	C1	C1		C1				62

Notes: B denotes a bid for a B/C block, C1 denotes a bid for a $C1/C2$ block, and D denotes a bid for a D/E block

Why did Bragg choose to bid on A blocks? In this auction, bidding for A blocks was the only way for nationwide bidders to acquire two blocks in a given region, making demand for

[29] The design of the Canadian 600 MHz auction moves away from any special treatment of eligibility-reducing rounds, by applying revealed preference symmetrically against all rounds.

them predictably high. Therefore, it was a relatively safe assumption for Bragg that: (1) its bids from Rounds 37 and 68 would never win; and (2) the clock prices of A blocks would be rising faster than the price of its actual demand. As such, the set of bidding opportunities available to Bragg in the supplementary round was strictly larger than if Bragg would have simply dropped blocks instead of going through A blocks.

An alternative explanation for Bragg's bidding is an attempt to exploit a weakness of the hybrid WARP/point-monotonic activity rule. The hybrid rule in the Canadian auction allowed bidders to violate point monotonicity when WARP inequalities from all prior eligibility-reducing rounds were satisfied. Bragg might have strategically altered its bidding in eligibility-reducing rounds to facilitate placing relaxed bids (i.e., bids that violate point monotonicity) in future rounds. Furthermore, Bragg actually placed a relaxed bid in Round 94 that was almost twice the size of its eligibility limit. The bid of Round 94 was made legal by Bragg's earlier strategic bidding.

Observe that the bid in Round 94 is clearly inconsistent with truthful bidding since Bragg was able to place the same bid, say, in Round 78, but decided not to do so. By Round 94, the prices of B/C blocks in both Manitoba (MB) and Saskatchewan (SK) had increased by \$4.2 and \$18.9 million, respectively, and only then did Bragg expand its demand to include these licenses. Bragg's bid in Round 94 had actual material consequences for two nationwide incumbents, Rogers and TELUS, increasing their final payments by \$9.15 million and \$25.7 million, respectively.

Moreover, this strategic bidding in eligibility-reducing rounds left Bragg essentially unconstrained during the rest of the auction. It was feasible for Bragg to revert to its initial demands at the very end of the auction by exploiting this weakness in the activity rule.

One-Way Cat Flaps: Everything Everywhere in the UK 4G Auction

Point monotonicity can interfere with truthful bidding by trapping bidders in "one-way cat flaps". The bidding history of Everything Everywhere (EE) in the UK 4G auction has such an appearance (see Table 6).

In Round 38, EE switched from bidding on eight C blocks (150 eligibility points each) to bidding on nine E blocks (1 eligibility point each) . In subsequent rounds, the clock price of E blocks rapidly increased, while the clock price of C blocks stayed constant at £92 million. By Round 45, the package consisting of nine E lots had become relatively more expensive than the package with eight C lots. If EE's bids from Rounds 37 and 38 were truthful, EE's truthful bid in rounds 45 to 56 would have been for eight C lots. However, EE was trapped, since reverting to C blocks required more eligibility points and was not possible under point monotonicity. As a result, the activity rule prevented useful price discovery and artificially inflated the clock price for E blocks. In fact, if EE had been allowed to switch back to eight C lots, the EE's final clock allocation might have been a lot closer to the actual auction outcome in which EE won one $A1$ lot, seven C lots and zero E lots.

Table 6: "One-way cat flap" for EE

Round	**Bid**	**Clock Prices (in mil)**		**Price Difference (in mil)**
		C	E	between $8C$ and $9E$
37	$8C$	£87.6	£1.194	£690
38	$9E$	£92	£1.43	£723.1
39 – 45	$9E$	£92	£1.72 $\rightarrow$ £5.11	£720.5 $\rightarrow$ £690
46 – 52	$9E$	£92	£6.4 $\rightarrow$ £24.4	£678.4 $\rightarrow$ £516.4

Excluding rounds 46 to 52 from EE's bidding history reduces the total GARP violation from £591.5 million to £288 million, with 97% of the GARP violation occurring in eligibility-preserving rounds.

8 Conclusion

Activity rules are a critical element of spectrum auctions and other modern dynamic auctions. With an activity rule that is badly suited to the environment, bidders might find themselves forced into bidding for less than their true demands, harming both the efficiency and revenues of the auction. And without any activity rule, bidders would find the private benefits of delayed bidding to be irresistible. This would eviscerate the social benefits of informative pricing and, in extreme cases, allow the auction to continue indefinitely.

It is tricky to select suitable point values for use in a point-monotonic activity rule—and misinformation abounds during the auction consultation process. Spectrum regulators and bidders understand at a theoretical level that a badly calibrated activity rule could favor certain bidders and invite considerable mischief. In our empirical section, we provide clear evidence of a bidder actually being trapped in a "one-way cat flap" in a major spectrum auction. That this occurred in the 2013 UK 4G Auction is unsurprising, given that the point ratio between two of the license categories was set by the regulator at 150:1, whereas the value ratio (as measured by final clock prices) was less than 4:1. However, the problem should be seen as more endemic; for any choice of points, there exist bidder values and a price trajectory making straightforward bidding impossible.

The analysis directs us inevitably toward activity rules derived from classical revealed preference. We introduce three axioms that formalize desirable properties of activity rules: an activity rule should (1) enforce the Law of Demand, (2) allow straightforward bidding, and (3) protect bidders from reaching dead ends. Our main result shows that the GARP activity rule is essentially the unique rule that satisfies all three axioms. Replacing our second axiom with a slightly weaker "limited straightforward bidding" axiom, we find that the SARP rule is the most restrictive activity rule that satisfies the axioms. In addition, the GARP activity rule supports price-justified updating and can be relaxed to handle common limitations such as the presence of budget constraints and provisional winnings.

Another advantage of activity rules based on revealed preference is that, unlike point-based activity rules, they do not require monotonic price paths. Observe that classical Walrasian tâtonnement not only increases the prices of overdemanded goods, but also decreases the prices of underdemanded goods. By contrast, most dynamic spectrum auctions to date have imposed ascending price paths. A GARP activity rule has the potential to enable novel auction designs that remove this limitation, coming closer to operationalizing the Walrasian auctioneer and thereby improving auction performance.

The subject matter of this article is not purely a theoretical development, but is also an interesting empirical application. Innovation, Science and Economic Development (ISED), Canada's spectrum regulator, has recently conducted its 600 MHz auction using a CCA format that incorporated a hybrid GARP/point-monotonic activity rule. The auction appears to have been highly successful, generating revenues of $3.47 billion while implementing a set-aside policy that reserved 30 of 70 MHz in the spectrum band for regional operators, rather than the three national service providers.[30] To our knowledge, this was the first spectrum auction worldwide to utilize a GARP-based activity rule (although several past spectrum auctions, including two other Canadian auctions, had utilized WARP-based activity rules), and the analysis of the bidding data (once it becomes publicly available) would be of substantial interest to the market design community. In any case, it is indeed both surprising and eminently satisfying that there is now a bridge directly linking Sydney Afriat's classic 1967 article with a state-of-the-art spectrum auction in 2019.

References

[1] Afriat, S., "The Construction of a Utility Function from Expenditure Data," *International Economic Review* 8(1) (1967), 67–77.

[2] Ausubel, L. M., "An Efficient Ascending-Bid Auction for Multiple Objects," *American Economic Review* 94(5) (2004), 1452–1475.

[30] `https://www.canada.ca/en/innovation-science-economic-development/news/2019/04/600-mhz-spectrum-auction--process-and-results.html`. The rules, including the hybrid GARP activity rule, are published in Innovation, Science and Economic Development Canada (2018), Section 7.2 and Annexes C, D and E. The authors disclose that they advised ISED in the design and implementation of the Canadian 600 MHz auction.

[3] Ausubel, L. M., and O. Baranov, "Market Design and the Evolution of the Combinatorial Clock Auction," *American Economic Review: Papers and Proceedings* 104(5) (2014), 446–451.

[4] Ausubel, L. M., and O. Baranov, "Iterative Vickrey Pricing in Dynamic Auctions," working paper, University of Maryland, 2019.

[5] Ausubel, L. M. and P. Cramton, "Auctioning Many Divisible Goods," *Journal of the European Economic Association* 2(2–3) (2004), 480–493.

[6] Ausubel, L. M., P. Cramton and P. Milgrom, "The Clock-Proxy Auction: A Practical Combinatorial Auction Design," in P. Cramton, Y. Shoham and R. Steinberg, eds., *Combinatorial Auctions* (Cambridge: MIT Press, 2006), 115–138.

[7] Che, Y., and I. Gale, "Standard Auctions with Financially Constrained Bidders," *Review of Economic Studies* 65(1) (1998), 1–21.

[8] Federal Communications Commission, *Broadband Personal Communications Services, Major Trading Area Licenses, Frequency Blocks A & B: Bidder's Information Package*, (Washington, DC: FCC, 1994) `https://www.fcc.gov/auction/4/releases`.

[9] Harsha, P., C. Barnhart, D. C. Parkes, and H. Zhang, "Strong activity rules for iterative combinatorial auctions," *Computers & Operations Research* 37(7) (2010), 1271–1284.

[10] Hatfield, J. W., and P. R. Milgrom, "Matching with Contracts," *American Economic Review* 95(4) (2005), 913–935.

[11] Hauser, D., "Activity Rules and Equilibria in the Combinatorial Clock Auction," honors thesis, Department of Economics, University of Maryland, 2012.

[12] Houthakker, H. S., "Revealed Preference and the Utility Function," *Economica* 17(66) (1950), 159–174.

[13] Innovation, Science and Economic Development Canada, *Technical, Policy and Licensing Framework for Spectrum in the 600 MHz Band* (Ottawa: ISED, 2018) `https://www.ic.gc.ca/eic/site/smt-gst.nsf/eng/sf11374.html`

[14] Mas-Colell, A., M. D. Whinston, and J. R. Green, *Microeconomic Theory* (Oxford: Oxford University Press, 1995).

[15] McAfee, R. P. and J. McMillan, "Analyzing the Airwaves Auction," *Journal of Economic Perspectives* 10(1) (1996), 159–175.

[16] McMillan J., "Selling Spectrum Rights," *Journal of Economic Perspectives* 8(3) (1994), 145–162.

[17] Milgrom, P. R., "Putting Auction Theory to Work: The Simultaneous Ascending Auction," *Journal of Political Economy* 108(2) (2000), 245–272.

[18] Milgrom, P. R., *Putting Auction Theory to Work* (Cambridge: Cambridge University Press, 2004).

[19] Milgrom, P. R. and R. J. Weber, "A Theory of Auctions and Competitive Bidding," *Econometrica* 50(5) (1982), 1089–1122.

[20] Pitchik, C. and A. Schotter, "Perfect Equilibria in Budget-Constrained Sequential Auctions: An Experimental Study," *RAND Journal of Economics* 19(3) (1988), 363–388.

[21] Rochet, J.-C., "A Necessary and Sufficient Condition for Rationalizability in a Quasi-Linear Context," *Journal of Mathematical Economics* 16(2) (1987), 191–200.

[22] Roth, A. E. and A. Ockenfels, "Last-Minute Bidding and the Rules for Ending Second-Price Auctions: Evidence from eBay and Amazon Auctions on the Internet," *American Economic Review* 92(4) (2002), 1093–1103.

[23] Samuelson, P. A., "A Note on the Pure Theory of Consumer's Behavior," *Economica* 5(17) (1938), 61–71.

[24] Varian, H. R., "Revealed Preference," in M. Szenberg, L. Ramrattan and A. A. Gottesman, eds., *Samuelsonian Economics and the Twenty-First Century* (Oxford: Oxford University Press, 2006), chapter 6.

[25] Walras, L., *Elements of Pure Economics; or, the Theory of Social Wealth* (Lausanne: Corbaz, 1874).

[26] Warshall, S., "A Theorem on Boolean Matrices," *Journal of the Association of Computing Machinery* 9 (1962), 11–12.

A Appendix - Proofs

PROOF OF THEOREM 2. (ii) implies (i). If $\{v^k\}_{k=1}^t$ satisfying (4.3) exists, then for any distinct choice of indexes $t_1, ..., t_s \in \{1, ..., t\}$, we have

$$\begin{array}{rcl} v^{t_1} - v^{t_s} & \geq & p^{t_1}\left[x^{t_1} - x^{t_s}\right] \\ v^{t_2} - v^{t_1} & \geq & p^{t_2}\left[x^{t_2} - x^{t_1}\right] \\ ... & ... & ... \\ v^{t_s} - v^{t_{s-1}} & \geq & p^{t_s}\left[x^{t_s} - x^{t_{s-1}}\right] \end{array}$$

and (4.2) follows by summing these inequities together.

(i) implies (ii). For each $k < t$, define

$$\lambda^k = \max_{\{t_1,...,t_{s-1}\}} \left\{p^{t_1}\left[x^{t_1} - x^k\right] + ... + p^{t_{s-1}}\left[x^{t_{s-1}} - x^{t_{s-2}}\right] + p^t\left[x^k - x^{t_{s-1}}\right]\right\}$$

where $t_1, ..., t_{s-1}$ is a distinct choice of indexes from $\{1, ..., t-1\}$. Note that $\lambda^k \geq 0$ for all $k < t$ by construction since one feasible sequence is $(t_1 = k,\ t_2 = t)$. Also set $\lambda^t = 0$. For each $k \leq t$, define $v^k = p^t x^k - \lambda^k$. To prove (4.3), we need to show that

$$\lambda^j \leq \lambda^i + (p^j - p^t)(x^i - x^j) \qquad \text{for all} \qquad i, j = 1, ..., t.$$

These inequalities are trivially satisfied for $j = t$. For $j < t$, pick a distinct set of indexes $t_1, ..., t_{s-1} \in \{1, ..., t-1\}$ such that

$$\lambda^j = p^{t_1}\left[x^{t_1} - x^j\right] + ... + p^{t_{s-1}}\left[x^{t_{s-1}} - x^{t_{s-2}}\right] + p^t\left[x^j - x^{t_{s-1}}\right]$$

By (4.2), this sequence can be picked such that $t_r \neq j$ for all $r = 1, ..., s-1$. Then

$$\begin{aligned} \lambda^j &= p^j\left[x^j - x^i\right] + p^{t_1}\left[x^{t_1} - x^j\right] + ... + p^{t_{s-1}}\left[x^{t_{s-1}} - x^{t_{s-2}}\right] + p^t\left[x^i - x^{t_{s-1}}\right] \\ &\quad + p^j\left[x^i - x^j\right] - p^t\left[x^i - x^j\right] \\ &\leq \lambda^i + (p^j - p^t)(x^i - x^j) \end{aligned}$$

□

PROOF OF THEOREM 3. The GARP activity rule satisfies Axiom 1 by construction. When a bidder bids straightforwardly according to value function $v(.)$, Axiom 2 follows from Theorem 2. For Axiom 3, note that there exists a set of numbers $\{\tilde{v}^k\}_{k=1}^{t-1}$ that rationalizes collection (p^k, x^k), $k = 1, ..., t-1$ by Theorem 2. Given p^t, find r such that

$$r \in \arg\max_{k \in \{1,...,t-1\}} \tilde{v}^k - p^t x^k$$

and set $x^t := x^r$ and $\tilde{v}^t := \tilde{v}^r$. By construction, the set of numbers $\{\tilde{v}^k\}_{k=1}^t$ solves (4.3). Then, by Theorem 2, collection (p^k, x^k), $k = 1, ..., t$ satisfies quasilinear GARP immediately implying Axiom 3 since $x^t \in \{x^1, ..., x^{t-1}\}$.

GARP is the strictest rule to satisfy Axioms 1, 2 and 3. Suppose that after a history (p^k, x^k), $k = 1, ..., t-1$, bundle $z \in X^t$ for the GARP activity rule, but $z \notin X^t$ for another activity rule under consideration. According to Theorem 2, there exists a value function $v(.)$ such that $z \in D(p^t, v)$ implying a violation of Axiom 2.

$\square$

PROOF OF THEOREM 4. The proof is carried out in two steps. At the first step, we provide an algorithm for constructing a minimal value function $\tilde{v}(\cdot)$ with two properties:

(1) For any bundle $z \in \Omega$, $v(z) \leq \tilde{v}(z) \leq v(z) + \delta$, and $\tilde{v}(x^t) = v(x^t) + \delta$;

(2) $\tilde{v}(\cdot)$ rationalizes the bidding history $(p^s,\, x^s)$ $\;s = 1, ..., t-1$.

At the second step, we prove that value function $\tilde{v}(\cdot)$ also rationalizes the demand choice made in round t.

STEP 1: Construct value function $\tilde{v}(\cdot)$ using the following algorithm:

S0: (Initialization) Set $A^0 = \{ x^t \}$, $B^0 = \{ z \in \Omega : z \in \{x^1, ..., x^{t-1}\} \;\&\; z \neq x^t \}$, $\tilde{v}^0(\cdot) = v(\cdot)$ and $D^0 = \delta$.

S1: (Loop) At step $k \geq 1$ of the algorithm:

S1.1: If $B^{k-1} \neq \emptyset$, calculate $\Delta(z)$ for each $z \in B^{k-1}$:

$$\Delta(z) = \min_{z' \in A^{k-1} \;\&\; s:\, x^s = z} \{\tilde{v}^{k-1}(z) - \tilde{v}^{k-1}(z') + p^s \cdot [z' - z]\} \,,$$

and calculate Δ^k and z^k:

$$\Delta^k = \min_{z \in B^{k-1}} \Delta(z) \qquad \text{and} \qquad z^k \in \arg\min_{z \in B^{k-1}} \Delta(z) \,.$$

Finally, set adjustment $d^k = \min \{ D^{k-1},\ \Delta^k \}$.

S1.2: If $B^{k-1} = \emptyset$, set adjustment $d^k = D^{k-1}$.

S1.3: Calculate $\tilde{v}^k(.)$ and D^k:

$$\tilde{v}^k(z) = \begin{cases} \tilde{v}^{k-1}(z) + d^k & z \in A^{k-1} \\ \tilde{v}^{k-1}(z) & z \notin A^{k-1} \end{cases} \qquad \text{and} \qquad D^k = D^{k-1} - d^k$$

S1.4: If $D^k > 0$, set $A^k = A^{k-1} \cup z^k$ and $B^k = B^{k-1} \setminus z^k$ and repeat from S1.1. If $D^k = 0$, go to S2.

S2: (Termination) Set $\tilde{v}(\cdot) = \tilde{v}^k(\cdot)$ and $A = A^k$ and terminate.

First, note that this algorithm runs in finite time since at each step of the loop one bundle is removed from set B^k and added to set A^k (once B^{k-1} is empty, the algorithm necessarily terminates after step k). Second, by construction, $\Delta(z) \geq 0$ for all $z \in B^{k-1}$ for any $k \geq 1$ and adjustment $d^k \geq 0$ at all steps of the algorithm (i.e., $\tilde{v}^k(\cdot) \geq \tilde{v}^{k-1}(\cdot)$) for any $k \geq 1$. Since the cumulative positive adjustment for any bundle $z \in \Omega$ never exceeds δ, $\tilde{v}(z) - v(z) \leq \delta$ for any $z \in \Omega$. Finally, $\tilde{v}(\cdot)$ rationalizes bidding history (p^s, x^s) $s = 1, ..., t-1$ since value function $v(\cdot)$ rationalizes bidding history (p^s, x^s) $s = 1, ..., t-1$ and, by construction, the algorithm never violates these revealed preference constraints.

STEP 2: Now we show that:

$$\tilde{v}(z) - p^t \cdot z \leq \tilde{v}(x^t) - p^t \cdot x^t \qquad \forall\, z \in \Omega.$$

First, note that $\tilde{v}(x^t) = v(x^t) + \delta$ and $\tilde{v}(z) = v(z)$ for any $z \in \Omega \setminus A$. Then, for any $z \in \Omega \setminus A$,

$$\begin{aligned} \tilde{v}(z) - p^t \cdot z &= v(z) - p^t \cdot z \leq \pi(p^t, v) = \delta + v(x^t) - p^t \cdot x^t \\ &= \tilde{v}(x^t) - p^t \cdot x^t \end{aligned}.$$

For any $z \in A$, there exists a distinct choice of indexes $t_1, ..., t_s$ such that $x^{t_1} = z$ and

$$\begin{aligned} \tilde{v}(x^{t_1}) - \tilde{v}(x^{t_2}) &= p^{t_1}[x^{t_1} - x^{t_2}] \\ \tilde{v}(x^{t_2}) - \tilde{v}(x^{t_3}) &= p^{t_2}[x^{t_2} - x^{t_3}] \\ \ldots \quad & \ldots \quad \ldots \\ \tilde{v}(x^{t_s}) - \tilde{v}(x^{t}) &= p^{t_s}[x^{t_s} - x^{t}] \end{aligned}$$

But then

$$\begin{aligned} \tilde{v}(z) - p^t \cdot z &= \tilde{v}(x^t) + p^{t_s}[x^{t_s} - x^t] + \ldots + p^{t_1}[x^{t_1} - x^{t_2}] - p^t \cdot z \\ &= \tilde{v}(x^t) - p^t \cdot x^t + \lambda \\ &\leq \tilde{v}(x^t) - p^t \cdot x^t \end{aligned}$$

where

$$\lambda = p^{t_s}[x^{t_s} - x^t] + \ldots + p^{t_1}[x^{t_1} - x^{t_2}] + p^t[x^t - x^{t_1}] \leq 0$$

since collection (p^s, x^s) $s = 1, ..., t$ satisfies GARP. Finally, in case $\tilde{v}(\vec{0}) > 0$, the value function can be normalized to yield $\tilde{v}(\vec{0}) = 0$ by subtracting $\tilde{v}(\vec{0})$ from all values. Note that the normalized value function would still rationalize (p^s, x^s) $s = 1, ..., t$ and $|\tilde{v}(z) - v(z)| \leq \delta$ for any $z \in \Omega$. □

PROOF OF THEOREM 5. In Theorem 3, we have established that the GARP activity rule is the strictest rule to satisfy Axioms 1, 2 and 3. Here we show that it is also the weakest rule to satisfy these axioms when the auctioneer can quote non-linear bundle prices. To prove this result, we consider an activity rule that is weaker than GARP. First, we show that for any

GARP violation we can construct $p^{t+1}(.)$ such that choosing any bundle in the corresponding GARP cycle violates the law of demand, so no bundle in the cycle is feasible by Axiom 1. Next, we set prices on other previously-demanded bundles high enough to ensure violations of the law of demand with respect to x^t. As a result, there are no previously-demanded bundles that are feasible at $p^{t+1}(.)$ implying a violation of Axiom 3.

Formally, for a nonlinear price trajectory $p^t(.)$, the quasilinear WARP condition (analog of (4.1)) is:

$$p^{t_1}(x^{t_1}) \;-\; p^{t_1}(x^{t_2}) \;+\; p^{t_2}(x^{t_2}) \;-\; p^{t_2}(x^{t_1}) \;\leq\; 0,$$

and the quasilinear GARP condition (analog of (4.2)) is:

$$p^{t_1}(x^{t_1}) - p^{t_1}(x^{t_s}) + p^{t_2}(x^{t_2}) - p^{t_2}(x^{t_1}) + ... + p^{t_s}(x^{t_s}) - p^{t_s}(x^{t_{s-1}}) \;\leq\; 0$$

Suppose that round t is the first round when collection $(p^k(.), x^k)$, $k = 1, ..., t$ satisfies WARP and does not satisfy GARP. Then there exists a distinct choice of indexes $t_1, ..., t_s \in \{1, ..., t\}$ where $s \geq 3$ and $t_s = t$ such that

$$p^{t_1}(x^{t_1}) - p^{t_1}(x^{t_s}) + p^{t_2}(x^{t_2}) - p^{t_2}(x^{t_1}) + ... + p^{t_s}(x^{t_s}) - p^{t_s}(x^{t_{s-1}}) \;=\; L \;>\; 0$$

For any price vector $p(.)$, the GARP violation can be restated as:

$$\begin{aligned}
0 \;<\; L \;\;=&\;\; p^{t_1}(x^{t_1}) - p^{t_1}(x^{t_s}) + p^{t_2}(x^{t_2}) - p^{t_2}(x^{t_1}) + ... + p^{t_s}(x^{t_s}) - p^{t_s}(x^{t_{s-1}}) \\
&- [\,p(x^{t_1}) - p(x^{t_s}) + p(x^{t_2}) - p(x^{t_1}) + ... + p(x^{t_s}) - p(x^{t_{s-1}})\,] \\
=&\;\;\;\; p(x^{t_s}) \;-\; p(x^{t_1}) \;+\; p^{t_1}(x^{t_1}) \;-\; p^{t_1}(x^{t_s}) \\
&+ p(x^{t_1}) \;-\; p(x^{t_2}) \;+\; p^{t_2}(x^{t_2}) \;-\; p^{t_2}(x^{t_1}) \\
&... \\
&+ p(x^{t_{s-1}}) \;-\; p(x^{t_s}) \;+\; p^{t_s}(x^{t_s}) \;-\; p^{t_s}(x^{t_{s-1}})
\end{aligned}$$

Then, a price vector $p^{t+1}(.)$ can be constructed as follows. For bundles $x^{t_1}, ..., x^{t_s}$, set $p^{t+1}(.)$ such that

$$\begin{aligned}
p^{t+1}(x^{t_s}) \;-\; p^{t+1}(x^{t_1}) \;+\; p^{t_1}(x^{t_1}) \;-\; p^{t_1}(x^{t_s}) \;\;&=\;\; L/s \;>\; 0 \\
p^{t+1}(x^{t_1}) \;-\; p^{t+1}(x^{t_2}) \;+\; p^{t_2}(x^{t_2}) \;-\; p^{t_2}(x^{t_1}) \;\;&=\;\; L/s \;>\; 0 \\
... \;\;&=\;\; ... \\
p^{t+1}(x^{t_{s-1}}) \;-\; p^{t+1}(x^{t_s}) \;+\; p^{t_s}(x^{t_s}) \;-\; p^{t_s}(x^{t_{s-1}}) \;\;&=\;\; L/s \;>\; 0
\end{aligned}$$

This is always feasible since the above system of linear equations has s equations and s unknowns. By construction, bundles $x^{t_1}, ..., x^{t_s}$ are infeasible in round $t+1$ since bidding for them would violate at least one WARP inequality. Next, given $p^{t+1}(x^t)$, for any other previously-demanded bundle z, set $p^{t+1}(z)$ such that

$$p^{t+1}(z) \;-\; p^{t+1}(x^t) \;+\; p^t(x^t) \;-\; p^t(z) \;=\; \epsilon \;>\; 0.$$

for some $\epsilon > 0$. By construction, all previously demanded bundles $\{x^1, ..., x^t\}$ are infeasible in round $t+1$ with prices $p^{t+1}(.)$ since bidding on any bundle causes at least one WARP violation.

$\square$

PROOF OF PROPOSITION 2. Part (a). For any distinct choice of indexes $t_1, ..., t_s \in \{1, ..., t-1\}$ and any bundle $z' \leq z$,

$$\begin{aligned} & p^{t_1}\left[x^{t_1} - z'\right] + p^{t_2}\left[x^{t_2} - x^{t_1}\right] + ... + p^{t_s}\left[x^{t_s} - x^{t_{s-1}}\right] + p^t\left[z' - x^{t_s}\right] \\ = \; & p^{t_1}\left[x^{t_1} - z\right] + p^{t_2}\left[x^{t_2} - x^{t_1}\right] + ... + p^{t_s}\left[x^{t_s} - x^{t_{s-1}}\right] + p^t\left[z - x^{t_s}\right] \\ & + (z - z')(p^{t_1} - p^t) \\ \leq \; & 0 \end{aligned}$$

since bundle $z \in X^t$ by assumption and $(z - z')(p^{t_1} - p^t) \leq 0$ due to non-decreasing prices. Thus, bundle $z' \in X^t$.

Part (b). Since $z \leq x^{t-1}$, then $z \in X^{t-1}$ by part (a). Then there exists a value function v, that rationalizes the bidding data, such that $z \in D(p^{t-1}, v)$. Since the clock price of bundle z is the same under p^{t-1} and p^t, and the clock price for any other bundle $z' \in \Omega$ weakly increased, $z \in D(p^t, v)$. Then $z \in X^t$ by Axiom 2.

$\square$

PROOF OF THEOREM 6. (ii) implies (i). For any distinct choice of indexes $t_1, ..., t_s \in \{1, ..., t\}$ such that $x^{t_1} \neq x^{t_s}$, (4.2) follows by Theorem 2. Suppose that this sequence violates (4.7), i.e.:

$$p^{t_1}\left[x^{t_1} - x^{t_s}\right] + p^{t_2}\left[x^{t_2} - x^{t_1}\right] + ... + p^{t_s}\left[x^{t_s} - x^{t_{s-1}}\right] = 0. \tag{A.1}$$

Then it must be the case that

$$\begin{array}{rcl} v^{t_1} - v^{t_s} & = & p^{t_1}\left[x^{t_1} - x^{t_s}\right] \\ v^{t_2} - v^{t_1} & = & p^{t_2}\left[x^{t_2} - x^{t_1}\right] \\ ... & ... & ... \\ v^{t_s} - v^{t_{s-1}} & = & p^{t_s}\left[x^{t_s} - x^{t_{s-1}}\right] \end{array} \tag{A.2}$$

and for the ranking order $r(.)$, we must have

$$r(x^{t_s}) > r(x^{t_1}) \geq r(x^{t_2}) \geq ... \geq r(x^{t_{s-1}}) \geq r(x^{t_s}) \tag{A.3}$$

which contradicts the existence of the ranking order $r(.)$.

(i) implies (ii). Since SARP condition (4.7) implies GARP condition (4.2), value function that rationalizes bidding history exists by Theorem 2. Suppose that the corresponding ranking order $r(.)$ which assigns a different integer number to all distinct bundles in $\{x^1, ..., x^t\}$ and satisfies

(4.9) does not exist. Then there must exist a sequence $t_1, ..., t_s \in \{1, ..., t\}$ where $x^{t_1} \neq x^{t_s}$ such that (A.2) holds (if there are no cycles, then a finitely many ties can be resolved by assigning a lower ranks to bundles that were demanded when the bidder was tied). But (A.2) implies (A.1) which violates (4.7). □

PROOF OF THEOREM 7. SARP satisfies Axioms 1, 2′ and 3. If bidder bids straightforwardly according to value function $v(.)$ and break ties (if applicable) using a ranking order $r(.)$, then Axiom 2′ follows from Theorem 6. SARP activity rule satisfies Axiom 1 by construction. For Axiom 3, note that there exits value function $\tilde{v}(.)$ and an order ranking $\tilde{r}(.)$ that rationalizes collection (p^k, x^k), $k = 1, ..., t-1$ by Theorem 6. By construction, any bundle $z \in \{x^1, ..., x^{t-1}\}$ that maximizes profit given $\tilde{v}(.)$ and p^t (and has the lowest rank out of all bundles that maximize profit according to $\tilde{r}(.)$) also belongs to X^t by Axiom 2′ (since the bidder might be bidding straightforwardly according to $\tilde{v}(.)$ and $\tilde{r}(.)$). Therefore, at least one prior demand is feasible in round t implying Axiom 3.

SARP is the most restrictive activity rule that satisfies Axioms 1, 2′ and 3. Suppose that after a history (p^k, x^k), $k = 1, ..., t-1$, bundle $z \in X^t$ for the SARP activity rule, but $z \notin X^t$ for the activity rule under consideration. Set $x^t = z$ and construct a value function as follows. First, for each $k \leq t$, define

$$\mu^k \;=\; \max_{\{t_1=t,\ldots,t_s=k\}} \left\{p^{t_1}\left[x^{t_1} - x^k\right] \;+\; ... \;+\; p^{t_{s-1}}\left[x^{t_{s-1}} - x^{t_{s-2}}\right] \;+\; p^{t_s}\left[x^{t_s} - x^{t_{s-1}}\right]\right\}$$

where $t_1 = t$, $t_s = k$ and $t_2, ..., t_{s-1}$ is a distinct choice of indexes from $\{1, ..., t\}$. Note that $\mu^k < 0$ for all k such that $x^k \neq x^t$ by (4.7) and $\mu^t = 0$. Second, for each $k \leq t$, define $v^k = p^t x^k + \mu^k$. To prove (4.8), we need to show that

$$\mu^i \;\leq\; \mu^j + (p^j \;-\; p^t)\,(x^i \;-\; x^j) \qquad \text{for all} \qquad i, j = 1, ..., t.$$

These inequalities are trivially satisfied when $j = t$ or $i = j$. For $j < t$, pick a distinct set of indexes $t_1, ..., t_s \in \{1, ..., t\}$ where $t_1 = t$ and $t_s = i$ such that

$$\mu^i \;=\; p^t\left[x^t - x^i\right] \;+\; ... \;+\; p^{t_{s-1}}\left[x^{t_{s-1}} - x^{t_{s-2}}\right] \;+\; p^i\left[x^i - x^{t_{s-1}}\right]$$

Then

$$\begin{aligned}
\mu^i \;&=\; p^t\left[x^t - x^j\right] + ... + p^{t_{s-1}}\left[x^{t_{s-1}} - x^{t_{s-2}}\right] + p^i\left[x^i - x^{t_{s-1}}\right] + p^j\left[x^j - x^i\right] \\
&\quad + p^j\left[x^i - x^j\right] \;-\; p^t\left[x^i - x^j\right] \\
&\leq\; \mu^j \;+\; (p^j \;-\; p^t)\,(x^i \;-\; x^j)
\end{aligned}$$

Therefore, $\{v^k\}_{k=1}^t$ rationalizes bidding history $(p^1, x^1), ..., (p^{t-1}, x^{t-1}), (p^t, x^t)$. Finally, note that for any k such that $x^k \neq x^t$, we have

$$v^k \;-\; p^t\,x^k \;=\; \mu^k \;<\; 0 \;=\; \mu^t \;=\; v^t \;-\; p^t\,x^t$$

and $x^t = z$ is the unique demand for value function $\{v^k\}_{k=1}^t$ at price p^t and the activity rule under consideration violates Axiom 2′. $\square$

PROOF OF PROPOSITION 3. For any distinct choice of indices $t_1, ..., t_s \in \{1, ..., t\}$, we have:

$$\begin{aligned} & \sum_{k=1}^{s-1} \left[mv^{t_{k+1}}(x^{t_s}, x^{t_k}) - mv^{t_k}(x^{t_s}, x^{t_k}) \right] \\ = \;& v^{t_1}(x^{t_1}) - v^{t_1}(x^{t_s}) + v^{t_2}(x^{t_2}) - v^{t_2}(x^{t_1}) \; + \; ... \; + \; v^{t_s}(x^{t_s}) - v^{t_s}(x^{t_{s-1}}) \\ \geq \;& p^{t_1}\left[x^{t_1} - x^{t_s}\right] \; + \; p^{t_2}\left[x^{t_2} - x^{t_1}\right] \; + \; ... \; + \; p^{t_s}\left[x^{t_s} - x^{t_{s-1}}\right] \\ = \;& \sum_{k=1}^{s-1} (p^{t_{k+1}} \; - \; p^{t_k})(x^{t_s} \; - \; x^{t_k}). \end{aligned}$$

$\square$

PROOF OF THEOREM 8. Suppose that a bidder bids straightforwardly according to value functions $v^1(\cdot), ..., v^{t-1}(\cdot)$ in rounds $1, ..., t-1$ and that the bidding history (p^1, x^1),...,(p^{t-1}, x^{t-1}) satisfies GARP. Suppose that bidding for bundle $z \in D(p^t, v^t(\cdot))$ violates the GARP activity rule. Then there exists a sequence $t_1, ..., t_s \in \{1, ..., t\}$ where $t_s = t$ such that

$$0 \; < \; \sum_{k=1}^{s-1} (p^{t_{k+1}} \; - \; p^{t_k}) \cdot (z \; - \; x^{t_k}) \; \leq \; \sum_{k=1}^{s-1} \left[mv^{t_{k+1}}(z, x^{t_k}) \; - \; mv^{t_k}(z, x^{t_k}) \right],$$

(where the last inequality follows from Proposition 3) contradicting (5.6). Thus, bidding for $z \in D(p^t, v^t(\cdot))$ in round t cannot violate the GARP activity rule. $\square$