VCG, the Core, and Assignment Stages in Auctions

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Abstract

The Vickrey-Clarke-Groves (VCG) mechanism is one of the most compelling constructs in mechanism design, but the presence of complementarities creates the theoretical possibility of non-core outcomes. In this paper, we empirically document such outcomes—including three zero-revenue rounds—in the Federal Communications Commission's Broadcast Incentive Auction. We also model the two-stage approach commonly used for auctioning spectrum, where bidders bid first for generic blocks and then bid for assignments of specific frequencies. We demonstrate that this approach may induce complementarities in the second stage even when the underlying preferences are well-behaved. Unexpectedly, assignment stages prove to be likely environments for non-core outcomes.

Keywords: VCG mechanism, core, spectrum auction, assignment stage

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1. Introduction

One of the most elegant and compelling constructs in the toolbox of mechanism design is the Vickrey-Clarke-Groves (VCG) mechanism.[1] In appropriate economic environments, asking agents to report their valuations for goods, allocating the goods efficiently relative to these reports, and compensating agents according to an opportunity cost calculation relative to these reports gives rise to an elegant mechanism. Truth-telling becomes a dominant strategy inducing efficiency.

However, almost two decades ago, economists added a cautionary amendment to our understanding.[2] Complementarities of goods might create complementarities of agents, causing VCG payments to lie outside the core (i.e., to be uncompetitively low). They may also make the mechanism vulnerable to a host of other anomalies. Two winning bidders may reduce their payments by colluding to increase each of their bids. Two losing bidders may convert themselves into winning bidders by coordinating on increasing their respective bids. A single agent may increase its payoff in the mechanism by placing some of its bids through a "shill" (i.e., a fictitious party).

A zero-revenue outcome, the most extreme ramification of this problem, becomes a theoretical possibility. Consider the "local-local-global" model, with two goods (A and B) and three bidders (1, 2 and 3) possessing the following valuations:[3]

$$\begin{array}{llll} v_1(AB)=2 & v_1(A)=0 & v_1(B)=0 & v_1(\varnothing)=0\,; \\ v_2(AB)=2 & v_2(A)=2 & v_2(B)=0 & v_2(\varnothing)=0\,; \\ v_3(AB)=2 & v_3(A)=0 & v_3(B)=2 & v_3(\varnothing)=0\,. \end{array}$$

The global bidder 1 views goods A and B as perfect complements and values only the AB combination at 2. Local bidder 1 values only good A, at 2; and local bidder 2 values only good B, also at 2. In this classic example, good A is awarded to bidder 2 and good B is awarded to bidder 3, since this is the value-maximizing allocation. However, the price paid by each of the local bidders in the VCG mechanism is zero: the social surplus with both local bidders present equals 4, while the social surplus with either local bidder absent equals 2, implying that each local bidder should keep the entire social surplus that it creates. As such, the VCG mechanism's revenues of zero are disappointing and the outcome is outside the core, since there is a blocking coalition (the seller and the global bidder) whose coalitional value equals 2.

Despite the theoretical literature, we are unaware of any documented empirical examples of zero-revenue outcomes. One possible reason for the dearth of known examples is that these observations spawned a literature on core-selecting mechanisms,[4] and real-world auctioneers, aware of this problem, typically include core adjustments in their rules. An alternative explanation for the lack of examples is that zero-revenue outcomes are merely a theoretical curiosity and not a practical problem.

[1] Vickrey (1961), Clarke (1971), Groves (1973).
[2] Ausubel and Milgrom (2002, 2006).
[3] Ausubel and Milgrom (2002) at pp. 5.
[4] Day and Raghavan (2007), Day and Milgrom (2008), Day and Cramton (2012).

In this paper, we document the occurrence of non-core outcomes (including zero-revenue outcomes) in what may initially appear to be an unlikely place: the assignment stage of a spectrum auction. Auctions of spectrum licenses that are close substitutes are often dichotomized into two phases. First, in the allocation stage, bidders bid for “generic” licenses (licenses that not attached to any specific frequency). Second, in the assignment stage, bidders express their preferences for different assignments of specific frequencies corresponding to their generic winnings from the allocation stage and the assignment that maximizes the sum of bidders’ values, as expressed by the bids, is selected.

It is initially easy to overlook that complementarities would even be present in assignment stages. The presence of complementarities in mobile telephony is typically driven by minimum scale and contiguity considerations. For example, LTE technology for mobile phones is able to achieve substantially greater data speeds by using a single segment of contiguous spectrum than by using an equivalent bandwidth of non-contiguous spectrum. Thus, in an auction of four 10 MHz spectrum blocks that are identified (in ascending order of frequencies) as the A, B, C and D blocks and are otherwise equivalent, a winner of two blocks in the allocation stage would wish to be assigned the “AB” combination, the “BC” combination, or the “CD” combination. Being fully aware of this issue, auctioneers typically restrict multiple block winnings to contiguous assignments. As a result, it might appear that all standard sources of complementarities have been either removed or contained within the allocation stage.

In this paper, we show why, contrary to initial expectations, an assignment stage is a very likely place to observe non-core outcomes.[5] To aid our narrative, we develop a stylized model of a two-stage Vickrey auction and introduce an equilibrium concept inspired by the dominant-strategy property of the VCG mechanism (Section 2).

Then, motivated by the well-known gross substitutes property,[6] we derive sufficient conditions for the equilibrium outcome of the two-stage Vickrey auction to be in the core with respect to equilibrium bids. In particular, we show that bidders’ preferences must exhibit *assignment substitutes* (a less restrictive version of gross substitutes) to prevent non-core outcomes in the assignment stage. However, in typical spectrum settings, bidder’s values exhibit *assignment complementarities* rather than substitutes due to contiguity considerations; and, ironically, separating an auction into two stages puts these complementarities under the spotlight and sets the stage for visible core violations in the assignment stage (Section 3).

We then document the occurrence of zero-revenue outcomes in the assignment stage of a very conspicuous, recent spectrum auction: the Federal Communication Commission (FCC) Broadcast Incentive Auction of 2016–17. In this auction, three rounds of the assignment stage produced zero-revenue outcomes and fully one-sixth of the assignment rounds yielded VCG outcomes that were

[5] Indeed, the exercise of looking for non-core outcomes in an assignment stage may initially remind the reader of the allegory about the drunk who searches for his lost money under a street light, rather than in the dark alley where he lost it, because the lighting is better. Are we performing this exercise for the Incentive Auction only because it is “under the street light,” i.e., because we have the data (and because it has received the most attention of any spectrum auction in the past decade)? The role of the second part of the paper is to show that the assignment stage of a spectrum auction is actually one of the more likely places to find non-core VCG outcomes.

[6] Gross Substitutes property was introduced by Kelso and Crawford (1982).

outside the core (Section 4).[7] Finally, we draw some implications for the design of future spectrum auctions (Section 5).

Before presenting our model and analysis, we provide some background and motivations for the use of the two-stage approach in spectrum auctions.

Background for the Two-Stage Approach in Spectrum Auctions

The critical stylized fact behind the use of two-stage approaches in auctioning spectrum is that bidders obtain value from receiving contiguous frequencies. In spectrum auctions of the past decade, the justification for this fact has been in the LTE technology that is part of the 4G standard for mobile phones. Quite simply, a mobile provider can obtain significantly greater data throughput from two adjacent 10 MHz frequency blocks than from two noncontiguous 10 MHz blocks. While this economy of scale as stated is tied specifically to LTE technology, the enhanced value of contiguity has been part of the conventional wisdom of spectrum valuation since the 1990s—and mobile providers continue to desire contiguous spectrum for deployment of the newest 5G technology.

Concurrently, spectrum regulators and stakeholders have exhibited a strong preference for dynamic auction formats over sealed-bid auction formats. A wide range of reasons have been given for this preference, including: the price discovery induced by a dynamic auction;[8] the greater transparency of a dynamic auction process; [9] the more aggressive bidding, arising from the lessened winner's curse, of a dynamic auction;[10] the reduced risk of being seen to overbid under the "second-pricing" that results from dynamic auctions;[11] and the cognitive simplicity of a dynamic auction over the corresponding sealed-bid auction.[12] Whether one accepts the reasons given or not, there is no mistaking that the vast preponderance of spectrum auctions worldwide utilize dynamic auction formats.

Integrating the desirability of contiguous frequencies into standard dynamic auction formats creates possibilities for mischief. For example, the popular simultaneous multiple round auction (SMRA) format, which is dynamic but just a single stage, does not support contiguity restrictions, enabling anti-competitive strategies such as when in one auction, "one bidder was attempting to assemble five adjoining licenses and the spoiler continued to bid on the middle license in a strategy that became known as 'giving the middle finger'."[13] When licenses are close substitutes, splitting the auction into two stages provides an elegant solution to this problem: limited to bids for generic blocks during the allocation stage and contiguous assignments in the assignment stage, bidders are unable to disrupt winnings of others by strategically positioning themselves within the band.

[7] If one accounts for the fact that the configurations of winnings in some of the rounds made non-core VCG outcomes impossible (for example, if there were only two winners), the reported fraction of rounds in which the VCG outcome was outside the core rises from 16.7% to 18.4%.

[8] Cramton (2013).

[9] Cite somebody.

[10] Milgrom and Weber (1982).

[11] Cite somebody.

[12] Cite Kagel?

[13] McAfee, McMillan, and Wilkie (2010), p. 181.

There are other practical disadvantages associated with one-stage dynamic auctions in settings with close substitutes. First, they frequently produce inordinately long auctions (with N licenses, it might take up to N rounds to increase the standing price on each license by just one price increment under the SMRA format).[14] Second, one-stage designs run a higher risk of inefficiency due to the fitting problem. Since bidders need to simultaneously discover both the "right" quantity allocation (constituting a larger portion of the auction value) and the "right" assignment (constituting a smaller portion of the auction value), a failure to coordinate on the assignment ("the fitting problem") might compromise discovering the "right" quantity winnings, and lead to significant welfare losses. In contrast, under the two-stage approach, the fitting problem never interferes with determining quantity allocation. Lastly, perhaps the strongest argument against the one-stage approach in settings with near substitutes is the unnecessary complexity of bidding decisions faced by bidders. However, it is challenging to formulate a good model capturing bidding complexity for the same reasons as it is to develop a good model of bounded rationality. In any case, the US FCC and the Australian Communications and Media Authority (ACMA), both long-time users of the one-stage approach, have adopted two-stage procedures for many of their recent auctions.

2. A Two-Stage Vickrey Auction

An auctioneer seeks to auction m indivisible blocks of spectrum denoted by set $M = \{1, ..., m\}$. The set of all possible combinations of blocks (i.e., packages) is denoted by $2^M = \{ z \mid z \subseteq M\}$. The auctioneer allocates these blocks to a set of bidders $N = \{1, ..., n\}$. Any restrictions on possible winnings imposed by the auctioneer (if any) are embedded into a set of admissible packages $\Omega \subseteq 2^M$.[15] For each bidder $i \in N$, the bidder's preferences over all packages in Ω are characterized by a value function $v_i(\cdot)$. The value of the null package, $\varnothing$, is normalized to zero. Assumptions about value functions and information structure are as follows:

(A1) *Pure Private Values:* Each bidder knows its own value for any package and this value is not affected by the values of other bidders;

(A2) *Quasilinear Utility:* Each bidder i's payoff from winning package z and making a payment of p is given by $v_i(z) - p$;

(A3) *Complete Information:* The value function $v_i(\cdot)$ of each bidder i is known to all bidders in N.

The set of all feasible assignments is given by:

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[14] The 2008 Canadian AWS auction extended 331 rounds and took 2 months. The 2014–15 US AWS-3 auction extended 441 rounds and took 2 ½ months.

[15] For example, the auctioneer can limit the number of blocks that can be won by individual bidders or restrict multi-block winnings to form a contiguous swath of spectrum. Note that the set of admissible packages Ω can be bidder-specific if needed.

$$X = \left\{ (x_1,...,x_n):\ x_i \in \Omega \quad \text{and} \quad x_i \cap x_j = \varnothing \quad \forall i, j \in N : i \neq j \right\}, \tag{3.1}$$

and the coalitional value function for bidders in coalition $C \subseteq N$ is given by:

$$w(C) = \max_{x \in X} \sum_{j \in C} v_j(x_j) \ . \tag{3.2}$$

Assignment $x = (x_1,...,x_n)$ is efficient if

$$\sum_{j \in N} v_j(x_j) = w(N). \tag{3.3}$$

A Vickrey outcome consists of an efficient assignment $x = (x_1,...,x_n)$ and a payment vector $p^V = (p_1^V,..., p_n^V)$ where

$$p_i^V = w(N \setminus \{i\}) - \sum_{N \setminus \{i\}} v_j(x_j), \tag{3.4}$$

and a core outcome consists of an efficient assignment $x = (x_1,...,x_n)$ and a payment vector $p^C = (p_1^C,..., p_n^C)$ belonging to the set of core payments $CP(x)$ given by:

$$CP(x) = \left\{ p^C \in \mathbb{R}_+^n :\ w(C) - \sum_{C} v_j(x_j) \le \sum_{N \setminus C} p_j^C \le \sum_{N \setminus C} v_j(x_j) \ \ \forall C \subseteq N \right\}. \tag{3.5}$$

Both the Vickrey outcome and core outcomes with respect to bids are defined in the same way by substituting bids for values.

When bidders view blocks in M as being "sufficiently close to each other" (in other words, each bidder mostly cares about the quantity of blocks it wins rather than the exact composition), the auctioneer can simplify the auction process by decomposing it in two sequential stages. We consider a two-stage procedure that uses a sequence of two Vickrey auctions to enable a direct compassion with the benchmark VCG mechanism. In the first stage (allocation stage), all blocks in M are treated as a homogeneous good and bidders bid on quantities. The auctioneer uses the Vickrey auction to determine the winning allocation of generic blocks and calculate corresponding payments.[16] In the second stage (assignment stage), the winners of the first stage compete for a specific assignment of generic blocks won in the first stage. Bidding in the second stage is optional since all winners of the first stage are necessarily assigned some blocks corresponding to their first-stage winnings. Once again, winning assignments and additional payments are determined by standard rules of the Vickrey auction.

Formally, in the allocation stage, each bidder i submits bid $b_i^I(q)$ for each quantity $q = 1,...,m$. Denote Y the set of all feasible quantity allocations, i.e.,

[16] The first stage could, instead, use a different auction format than the VCG mechanism, e.g., a uniform-price ascending clock auction. However, our analysis uses Vickrey auction for each stage, since it seems to give the fairest comparison between the two-stage process and either a one-stage process or the efficient outcome.

$$Y = \left\{ (y_1,\ldots,y_n):\ y_i \in \{0,\ldots,m\} \quad \forall i \in N \quad \text{and} \quad \sum_{j \in N} y_j \le m \right\}. \tag{3.6}$$

The auctioneer establishes the winning allocation of generic blocks y^* by solving

$$y^* \in Y^* = \arg\max \sum_{j \in N} b_j^I(y_j) \qquad \text{s.t.} \qquad y \in Y. \tag{3.7}$$

If Y^* contains multiple solutions, the auctioneer selects y^* according to a pre-determined tie-breaking procedure.[17] Denote $\Omega(q)$ the set of all assignment options that are consistent with quantity q, i.e.,

$$\Omega(q) = \left\{ (z_1,\ldots,z_m) \in \Omega:\ \sum_{k=1}^{m} z_k = q \right\}. \tag{3.8}$$

In the assignment stage, each bidder i submits bid $b_i^{II}(z)$ for each assignment option $z \in \Omega(y_i^*)$. Denote $X(y)$ the set of all feasible assignments that correspond to the quantity allocation y, i.e.,

$$X(y) = \left\{ (x_1,\ldots,x_n):\ x_i \in \Omega(y_i) \quad \text{and} \quad x_i \cap x_j = \varnothing \quad \forall i, j \in N : i \ne j \right\}. \tag{3.9}$$

The auctioneer selects the winning assignment x^* by solving

$$x^* \in X^* = \arg\max \sum_{j \in N} b_j^{II}(x_j) \qquad \text{s.t.} \qquad x \in X(y^*). \tag{3.10}$$

Again, if X^* contains multiple solutions, the auctioneer selects x^* according to a pre-determined tie-breaking protocol. Ultimately, bidder i wins package x_i^* and pays the sum of two Vickrey payments, one from the allocation stage and another one from the assignment stage.

Truthful Bidding and Truthful Subgame Perfect Equilibrium

Next we define a notion of truthful bidding in the two-stage Vickrey auction. For bidder i, denote $\underline{v}_i(q)$ and $\overline{v}_i(q)$ the values of the least preferred and the most preferred assignment options corresponding to q generic blocks, i.e.,

$$\underline{v}_i(q) = \min_{z \in \Omega(q)} v_i(z) \qquad \text{and} \qquad \overline{v}_i(q) = \max_{z \in \Omega(q)} v_i(z). \tag{3.11}$$

The truthful bidding strategy in the second stage is trivial. The Vickrey payment rule incentivizes bidders to bid truthfully on all feasible assignment options. For any given allocation of generic blocks $y = (y_1,\ldots,y_n)$, truthful bidding for bidder i consists of bidding zero for its least preferred assignment

[17] The tie-breaking procedure can have random components, but all random numbers must be drawn before the auction and disclosed to bidders such that they can perfectly predict an outcome of any tie-breaking situation.

option (since the bidder is guaranteed some assignment) and bidding its true marginal value on any other available option, i.e.,

$$b_i^{II}(z) = v_i(z) - \underline{v}_i(y_i) \qquad \forall z \in \Omega(y_i). \tag{3.12}$$

For each quantity allocation $y \in Y$, let $x_i(y)$ and $p_i(y)$ denote the assignment and payment, respectively, of bidder i in the second stage when all bidders bid truthfully according to (3.12).

For the first stage, a naive truthful bidding strategy consists of bidding based on the minimum value of the generic blocks $\underline{v}_i(q)$. However, this strategy is too conservative and causes an "underbidding regret" when bidder i is sure to win its preferred blocks in the second stage. Instead, a bidder should bid its "true" value of winning q generic blocks – the value that depends on opponents' bids through the ultimate assignment the bidder would get and the additional payment the bidder would have to make in the second stage.[18] Note that this dependence creates a major obstacle for analyzing this model under incomplete information, but not for the complete information treatment of this paper.

Denote $Y_i(q) = \{y \in Y : y_i = q\}$ the set of all feasible allocations in which bidder i allocated q generic blocks, and let $\mu_i(q) \in Y_i(q)$ denote bidder i's belief about the generic winnings profile after the first stage in which bidder i wins q generic blocks. Bidder i is said to bid truthfully in the first stage if it bids its "true" value of winning q generic blocks that incorporates the outcome of the assignment stage (assuming truthful bidding in the second stage) that would take place according to its belief $\mu_i(q)$:

$$b_i^{I}(q) = v_i(x_i(y)) - p_i(y) \qquad \text{where} \qquad y = \mu_i(q) \qquad \forall q = 1,\ldots,m. \tag{3.13}$$

Note that $b_i^I(q) \in [\, \underline{v}_i(q),\ \overline{v}_i(q)\,]$ in (3.13) by construction.

In equilibrium, bidders' beliefs about which assignment rounds take place must be consistent with the first-stage bids.

Definition 1. A bid-belief profile $\left\{ b_i^I(\cdot), b_i^{II}(\cdot), \mu_i(\cdot) \right\}_{i=1}^{n}$ forms a *truthful subgame perfect equilibrium (TSPE)* of the two-stage Vickrey auction if

a) for any quantity allocation $y \in Y$ after the first stage, each bidder i bids truthfully in the second stage (according to (3.12));
b) based on its belief $\mu_i(\cdot)$, each bidder i bids truthfully in the first stage (according to (3.13));
c) belief $\mu_i(\cdot)$ for each bidder i is consistent given the first-stage bids of its opponents, i.e.,

$$\mu_i(q) = y^*(y_i = q) \qquad \forall q = 1,\ldots,m, \tag{3.14}$$

[18] The first-stage bidding depends on (1) the profile of generic winnings after the first stage; and (2) the assignment bids that will be placed by bidders. [MAYBE WE CAN REMOVE THIS FOOTNOTE]

where $y^*(y_i = q)$ is the solution to the optimization problem (3.7) with an additional constraint restricting $y_i = q$.

We illustrate the TSPE concept with an example presented in Table 1. In this example, there are two blocks (A and B) and two bidders (1 and 2). In Stage 1, both bidders bid their true values for two generic blocks since winning two blocks unambiguously means winning blocks A and B. Deciding what to bid for one generic block is more involved since each bidder has to predict which block, A or B, it will ultimately get and at what price. When both bidders win one block each, Bidder 1 predicts that it will get block A and pay 5 in the assignment stage, and Bidder 2 anticipates winning block B and paying 0. As a result, Bidder 1 bids $b_1^I(1) = v_1(A) - 5 = 40$ and Bidder 2 bids $b_2^I(1) = v_2(B) - 0 = 25$.

Table 1: Example with Unique Truthful Subgame Perfect Equilibrium

	Bidder 1	*Bidder 2*
Values	$v_1(A) = 45,\ v_1(B) = 39$	$v_2(A) = 30,\ v_2(B) = 25$
	$v_1(AB) = 60$	$v_2(AB) = 50$
VCG Outcome	$(A,\ 25)$	$(B,\ 15)$

Unique TSPE of the Two-Stage Vickrey Auction

Beliefs	$\mu_1(1) = (1,1),\ \mu_1(2) = (2,0)$	$\mu_2(1) = (1,1),\ \mu_2(2) = (0,2)$
Bids in Stage 1	$b_1^I(1) = 40,\ b_1^I(2) = 60$	$b_2^I(1) = 25,\ b_2^I(2) = 50$
Outcome of Stage 1	$(y_1 = 1,\ 25)$	$(y_2 = 1,\ 20)$
Bids in Stage 2	$b_1^{II}(A) = 6$	$b_2^{II}(A) = 5$
TSPE Outcome	$(A,\ 30)$	$(B,\ 20)$

It follows that in the unique TSPE, Bidder 1 wins block A and pays 30 (25 + 5) in total, and Bidder 2 wins block B and pays 20 (20 + 0) in total. Observe that while the equilibrium assignment is efficient, the equilibrium revenue is higher than the revenue generated by the one-stage VCG mechanism. The revenue difference between two auction formats is not surprising. While they both use the Vickrey pricing principle, the counterfactual calculations performed by each auction are distinct.

Our first result can be framed as follows. The one-stage VCG mechanism always results in the efficient assignment, and can produce a non-core outcome only when there are complementarities among bidders. Under the two-stage approach, the scope for non-core outcomes becomes larger: a TSPE assignment is not guaranteed to be efficient, and when efficient, the corresponding payments may be too low to be in the core regardless of any complementarities among bidders (see example in Table A1 in the Appendix).

Proposition 1. Under assumptions (A1)—(A3), a TSPE of the two-stage Vickrey auction (if exists) can be inefficient. When a TSPE is efficient, the associated revenues can be higher/lower than the revenues of the one-stage VCG mechanism.

Under assumptions (A1)—(A3), a TSPE is subject to a standard assortment of equilibrium problems including non-existence, non-uniqueness and inefficiency (see examples in Table A1 and Table A2 in the Appendix). The culprit is straightforward — splitting the auction process into two sequential stages cannot work well unless bidders' preferences exhibit sufficient degree of homogeneity over licenses such that allocation of generic blocks has the first-order importance over assignments of specific blocks. The full analysis of existence and uniqueness of a TSPE in the two-stage Vickrey auction is beyond the scope of the paper. For our purposes, it is sufficient to make additional assumptions that guarantee existence of the unique and efficient TSPE. We utilize the following definition:

Definition. Allocation $y \in Y$ *value-dominates* another allocation $y' \in Y$ if

$$\sum_{\Delta(y,y')} \underline{v}_j(y_j) \;>\; \sum_{\Delta(y,y')} \overline{v}_j(y'_j), \tag{3.15}$$

where $\Delta(y,y') = \{\, j \in N : y_j \neq y'_j \,\}$.

Intuitively, allocation y *value-dominates* allocation y' when the aggregate value of the least preferred assignment options corresponding to y is greater than the aggregate value of the most preferred assignment options corresponding to y', ignoring bidders with the same number of generic blocks in both allocations. It immediately follows that any assignment $x \in X(y')$ is inefficient when allocation y' is value-dominated by some other allocation y, provided bidders winning the same number of blocks in both y and y' can be assigned the same frequencies.[19]

We make the following additional assumptions on the full collection of value functions $\{\, v_i(\cdot) \,\}_{i=1}^{n}$:

(A4) For each bidder $i \in N$ and each quantity $q = 1,\dots,m$, there exists allocation $\tilde{y} \in Y_i(q)$ that value-dominates all other allocations in $Y_i(q)$.

(A5) There exists allocation $y^* \in Y$ that value-dominates all other allocations in Y.

To motivate assumption (A4), consider a reduced game in which each bidder i bids such that $b_i^I(q) \in [\, \underline{v}_i(q),\ \overline{v}_i(q) \,]$ for all $q = 1,\dots,m$ in the first stage and plays its dominant strategy in the second stage.[20] In this reduced game, assumption (A4) guarantees that each bidder can predict the exact configuration of the assignment round that will take place after the bidder wins q generic blocks in the first stage. Consequently, there exists only one set of beliefs $\{\, \mu_i(\cdot) \,\}_{i=1}^{n}$ that is consistent with any collection of first-stage bids, and truthful bidding based on these beliefs forms the unique TSPE of the two-stage Vickrey auction. In this spirit, the unique TSPE of the two-stage Vickrey auction is comparable

[19] The contiguity restriction is one example of a constraint that can prevent the auctioneer from being able to assign the same blocks to bidders winning the identical number of generic blocks in allocations y and y'.

[20] Bidder i has a best response satisfying $b_i^I(q) \in [\, \underline{v}_i(q),\ \overline{v}_i(q) \,]$ for all q. However, bid profiles such that $b_i^I(q) \notin [\, \underline{v}_i(q),\ \overline{v}_i(q) \,]$ for some q are not necessarily dominated.

to the unique dominant strategy equilibrium of the (one-stage) VCG mechanism. Similarly, assumption (A5) guarantees that bids placed in the first stage of a reduced game always result in the efficient allocation of generic blocks.

Proposition 2. Given assumptions (A1)—(A4), the TSPE in the two-stage Vickrey auction exists and it is unique. When assumption (A5) is also satisfied, the equilibrium assignment is efficient.

Proof: In the Appendix.

3. Visible Non-Core Outcomes

The two-stage Vickrey auction does not provide any core guarantees with respect to values (Proposition 1). In this section we explore the core guarantees with respect to equilibrium bids. Suppose that there is no contiguity requirement and the set of admissible packages Ω is unconstrained (i.e., $\Omega = 2^M$). The well-known sufficient condition for the outcome of the (one-stage) VCG mechanism to be in the core is bidders' preferences satisfying the gross substitution.

Definition 2. Bidder i's value function $v_i(.)$ satisfies *gross substitutes (GS)* if for any two price vectors p and p' such that $p' \geq p$, and any $z \in D_i(p)$, there exists $z' \in D_i(p')$ such that $z'_k \geq z_k$ for all $k \in M$ such that $p_k = p'_k$ (where $D_i(p) = \arg\max_{z\in\Omega}\{ v_i(z) - p \cdot z \}$ denotes the demand correspondence of bidder i at price vector p).

Consider the allocation stage of the two-stage Vickrey auction. In the homogeneous setting, the GS condition reduces to *decreasing marginal values*. Then, to obtain core guarantees, the corresponding TSPE bids for each bidder i must also exhibit decreasing marginal values for all quantities, i.e., $b_i^I(q) - b_i^I(q-1) \geq b_i^I(q+1) - b_i^I(q)$ for any quantity q. Since in equilibrium $b_i^I(q) \in [\, \underline{v}_i(q), \overline{v}_i(q)\,]$, such bid relationship is implied by a value function that exhibits decreasing marginal values under any assignment.

Definition 3. Bidder i's value function $v_i(.)$ satisfies *assignment-independent decreasing marginal values (AIDMV)* if

$$\underline{v}_i(q) - \overline{v}_i(q-1) \geq \overline{v}_i(q+1) - \underline{v}_i(q) \qquad \forall\, q = 1,\ldots,m-1. \tag{4.1}$$

Proposition 3. Suppose that $\Omega = 2^M$ and the collection of value functions satisfies (A1)—(A5) and AIDMR. Then the Vickrey outcome of the allocation stage is in the core with respect to TSPE bids.

Note that the AIDMV property implies standard decreasing marginal values (i.e., the *submodularity* of a value function), but it does not imply (nor it is implied by) gross substitution. It follows that preferences satisfying GS do not provide core guarantees in the allocation stage (see example in Table A3 of the Appendix).

Next we consider the assignment stage. Since bidder's bids in the assignment stage are restricted to bundles with the constant number of blocks (corresponding to the generic winnings from the allocation stage), a weaker version of gross substitutes, as limited to bundles of the same size, is sufficient.

Definition 4. Bidder i's value function $v_i(.)$ satisfies *assignment substitutes* for quantity q if for any two price vectors p and p' such that $p' \geq p$, and any $z \in D_i(p \mid q)$, there exists $z' \in D_i(p' \mid q)$ such that $z'_k \geq z_k$ for all $k \in M$ such that $p_k = p'_k$ (where $D_i(p \mid q) = \arg\max_{z \in \Omega(q)} \{ v_i(z) - p \cdot z \}$ denotes the demand correspondence of bidder i at price vector p when limited to bundles in $\Omega(q)$).

Since an assignment stage is a continuation game after the allocation stage, we state our next proposition under weaker assumptions on bidders' values and the mechanism being used for the allocation stage.

Proposition 4. Suppose that $\Omega = 2^M$ and value functions satisfy (A1)—(A2) and *assignment substitutes* for each quantity $q = 1,...,m$. For any mechanism used for the allocation stage, the Vickrey outcome of every possible assignment stage is in the core with respect to truthful bids.

Proof: In the Appendix.

Note that gross substitutes imply assignment substitutes for all quantities, but the reverse is obviously false.[21] Thus, when bidders can bid on any packages (including non-contiguous packages) and their values satisfy gross substitution, the Vickrey outcome of any assignment stage must be in the core with respect to truthful bids.

Observe that Proposition 4 is not applicable to spectrum domains in which bidders value contiguous assignments. To see this, consider a bidder who wants two contiguous blocks (out of A, B, C or D) and does not care which ones. If a bidder initially demands AB, its demand will switch to CD once the price of B goes up, causing a demand drop for block A — an obvious violation of assignment substitutes. Thus, the desirability of contiguous assignments, one of the main rationales for utilizing the two-stage approach in spectrum auctions, is fundamentally incompatible with core guarantees in the assignment stage.

Next we show that the two-stage approach by itself can create core violations. Often, small assignment complementarities are "swamped" by bidders' values for quantities, ensuring that the there are no complementarities among bidders and the outcome of the (one-stage) VCG mechanism is in the core. In contrast, the same small complementarities become focal once the assignment is determined in isolation under the two-stage approach. We illustrate this point with example in

Table 2 which satisfies assumptions (A1)—(A5) to guarantee efficiency and uniqueness of the TSPE. In this example, there are four blocks (A, B, C and D) and three bidders with demand for at most two blocks. The outcome of the (one-stage) VCG mechanism is safely in the core since all two-winner core constraints are far from being binding:

$$p_1^V + p_2^V = 150 \geq 20, \qquad p_1^V + p_3^V = 210 \geq 80, \qquad p_2^V + p_3^V = 160 \geq 30.$$

[21] This follows from Corollary 1.3 in Murota and Shioura (2018).

Table 2: A Non-core Outcome in the Assignment Stage

	Bidder 1	*Bidder 2*	*Bidder 3*
Values*	$\underline{v}_1(1) = 150$	$\underline{v}_2(1) = 100$	$\underline{v}_3(1) = 300$
	$v_1(A) = 170$	$v_2(B) = 120$	$\underline{v}_3(2) = 500$
	$\underline{v}_1(2) = 200$	$\underline{v}_2(2) = 200$	$v_3(AB) = 520$
VCG Outcome	$(A,\ 100)$	$(B,\ 50)$	$(CD,\ 110)$

Unique TSPE of the Two-Stage Vickrey Auction

Beliefs	$\mu_1(1) = (1,1,2)$	$\mu_2(1) = (1,1,2)$	$\mu_3(1) = (1,2,1)$
	$\mu_1(2) = (2,0,2)$	$\mu_2(2) = (0,2,2)$	$\mu_3(2) = (1,1,2)$
Bids in Stage 1	$b_1^I(1) = 170$	$b_2^I(1) = 120$	$b_3^I(1) = 300$
	$b_1^I(2) = 200$	$b_2^I(2) = 200$	$b_3^I(2) = 500$
Outcome of Stage 1	$(y_1 = 1,\ 80)$	$(y_2 = 1,\ 30)$	$(y_3 = 2,\ 110)$
Bids in Stage 2	$b_1^{II}(A) = 20$	$b_2^{II}(B) = 20$	$b_3^{II}(AB) = 20$
TSPE Outcome	$(A,\ 80)$	$(B,\ 30)$	$(CD,\ 110)$

* – for Bidder 1, value for A block is 170, value for any other block (B, C or D) is 150 and value for any 2 blocks is 200. Values for Bidders 2 and 3 are interpreted analogously.

However, with the two-stage auction, the outcome of the assignment stage is outside the core. In the allocation stage, Bidders 1 and 2 win one block each, and Bidder 3 wins 2 blocks. In the assignment stage, Bidder 1 bids 20 for A, Bidder 2 bids 20 for B, and Bidder 3 bids 20 for AB. As a result, Bidders 1 and 2 are now complements since they are jointly bidding against Bidder 3. The result of this complementarity is quite dismal for the auctioneer and Bidder 3: Bidders 1 and 2 win their preferred assignments and pay zero—an obvious core violation given the bid of Bidder 3!

In practice, there are a number of reasons why observing non-core outcomes in the first stage is unlikely. First, a typical spectrum auctioneer allocates generic blocks via some dynamic auction format with a pricing rule other than the VCG. Second, a non-core outcome requires that in the absence of two or more winners, other bidders are able to acquire significantly larger amounts of spectrum. However, most spectrum auctions prevent bidders from buying large amounts of spectrum for competition reasons (e.g., "spectrum caps"), reducing the likelihood of outcomes outside the core (as in

Table 2 where bidders' demands are limited to two blocks). In contrast, assignment stages are routinely implemented as sealed-bid auctions with pricing rules based on the opportunity costs principle (either the VCG mechanism or a core-selecting auction such as "Nearest-VCG"). Since opportunity costs in the assignment stage are coming from obtaining "better blocks" rather than obtaining "more blocks", spectrum caps have no effect on the likelihood of non-core outcomes.

4. The Assignment Stage in the FCC's Broadcast Incentive Auction

The Broadcast Incentive Auction included two separate but interdependent auctions: a reverse auction, where television broadcasters bid to relinquish their spectrum usage rights, and a forward auction, where wireless operators bid to acquire freed-up spectrum.[22] For the forward auction, the FCC adopted a two-stage approach. In the first stage, the FCC used an ascending clock auction to award seven generic blocks of spectrum to bidders in 416 distinct partial economic areas (PEAS). In the second stage, bidders bid for physical frequency assignments in each of these regions.

The assignment stage was organized as a sequence of sealed-bid auctions, in six parallel sessions, where bidders bid for their assignments independently in each PEA, in descending order of population. To accelerate the process and to enhance geographic contiguity, PEAs with the same winners and winnings were consolidated into assignment regions.[23] As a result, the total number of assignment rounds was reduced from 416 to 228. Participating bidders were informed about their own assignments and payments in each region before bidding for their assignments in the next region.

The FCC adopted the VCG mechanism for the assignment stage. Each winner of generic blocks was guaranteed a contiguous assignment within the region. Bidders were invited to bid on all possible contiguous assignments corresponding to their generic winnings, even though some of these assignments would be incompatible with maintaining contiguity for other winners. This was done intentionally to avoid disclosing information about the winnings of other bidders.

The bidding data from the assignment stage of the Incentive Auction presents us with a unique opportunity to evaluate the extent of the core problem of the VCG mechanism. Treating bids as true values, we find 38 instances (out of 228) where VCG payments were too low to be in the core. The cumulative revenue shortage across all assignment rounds is $4,411,699. That is, assuming the same set of bids, if the FCC had used a core-selecting payment rule, the assignment stage gross revenues would have been $140,342,331 instead of $135,930,632 (about 3.25% higher). However, given that core-selecting auctions are not incentive-compatible in general (i.e., bidders can gain by bidding in a non-truthful manner), the implications of this comparison are limited.

Most notably, there were three instances (out of the 38 instances with core violations) where bidders literally paid zero! To the best of our knowledge, this was the first time that the "zero revenue" scenario of the VCG mechanism was documented in the field. Next we provide details for one of these instances.

The assignment of seven blocks in PEAs 224 and 287 (De Kalb, IL and Kenosha, WI) was made in assignment round 37. In this consolidated region, Dish Network won one generic block, while T-Mobile and U.S. Cellular won three generic blocks each. All bids, including discarded incompatible bids, submitted by bidders in this round are reported in Table 3, with winning bids displayed in bold.

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[22] A detailed description of the Incentive auction can be found in Aperjis, Ausubel, and Baranov (2020). All publicly-available bidding data can be found at https://auctiondata.fcc.gov/public/projects/1000.

[23] When several PEAS had the same winners and generic winnings, they were merged into a single region for the purposes of assignment. In the assignment stage, bidders were guaranteed the same assignment in all merged PEAS.

Table 3: Assignment Bids for PEAs 224 and 287 in $ million

Band Plan

A	B	C	D	E	F	G

Bidders (generic winnings)	Dish Network (1 block)	T-Mobile (3 blocks)	U.S. Cellular (3 blocks)
Compatible Bids (block(s), bid)	**A, 0**	ABC, 0.4m	ABC, 0
	D, 0	**BCD, 3.5m**	BCD, 0
	G, 0.237m	DEF, 0.010m	DEF, 4m
		EFG, 0	**EFG, 8.7m**
Incompatible Bids (block(s), bid)	B, 0.308m	CDE, 2.1m	CDE, 0.010m
	C, 0		
	E, 0		
	F, 0.310m		
VCG payment	**0**	**0**	**0**

The mechanics of the VCG payment calculations for this example are the same as in the example of the Introduction. U.S. Cellular's bid of $8.7 million for blocks EFG prevented Dish Network from winning block G. Independently, T-Mobile's bid of $3.5 million for blocks BCD also prevented Dish Network from winning block G because winning it would have forced U.S. Cellular to a noncontiguous assignment. Therefore, in the absence of either T-Mobile or U.S. Cellular, Dish Network would still have been assigned block A. As a result, the VCG payments of both T-Mobile and U.S Cellular equaled zero! In contrast, a core-selecting rule would have increased the joint payment of T-Mobile and U.S. Cellular to $237,000 and avoided the zero-revenue scenario.

Given that the problem of non-core outcomes was well known, the reader may wonder why the FCC adopted a VCG rule instead of a core-selecting rule for the assignment stage. The explanation is specific to this auction. In order for spectrum to be cleared in the Broadcast Incentive Auction, the revenues from the forward auction were required to exceed the costs from the reverse auction. This balanced budget constraint was based solely on the first, allocation stage and could not include revenues from the second, assignment stage (which had not yet occurred). Any higher payment rule in the assignment stage could induce lower bids and, as a result, lower revenues in the allocation stage. In turn, any diversion of revenues from the allocation stage created the risk of clearing less spectrum. To mitigate this risk, the lowest coherent payment rule for the assignment stage was adopted.

5. Conclusion

The theoretical possibility for the VCG mechanism to generate an outcome outside the core has been known for almost 20 years. One of the most extreme ramifications of this problem is the possibility of zero-revenue outcomes. That is, in a competitive auction where all bidders (winners and losers alike) make nontrivial bids, winners pay zero to secure their allocations. This counterintuitive anomaly violates fundamental auction principles and leads to a host of problematic issues. In this paper, we document zero-revenue outcomes in the field for the first time: in the assignment stage of the FCC's Broadcast Incentive Auction.

The majority of recent spectrum auctions for near substitutes allocate them in two sequential phases: an allocation stage followed by an assignment stage. In this paper, we introduce a stylized model of a two-stage Vickrey auction and solve for its truthful equilibrium. Our findings are twofold. First, revenues in the efficient equilibrium of the two-stage Vickrey auction are generally different from the revenues generated by the VCG mechanism. As a result, even when outcomes of both stages are in the core with respect to bids, the outcome of the two-stage Vickrey auction might be outside the core with respect to true values.

Second, to provide core guarantees in the assignment stage with respect to truthful bids, bidders must view all assignments corresponding to the same generic quantities as substitutes, a condition that never holds in the realm of spectrum auctions due to the general desirability of contiguous assignments. We show that the mere decomposition of the auction into two sequential stages brings extra focus on the unavoidable assignment complementarities and can result in non-core outcomes in the assignment stage. Thus, assignment stages turn up to be a very likely place for observing non-core outcomes in the Vickrey auction.

Just because the two-stage procedure potentially increases the likelihood of non-core outcomes and because we have observed non-core outcomes in the assignment stage does not mean that it should be abandoned. Quite the opposite, there are numerous practical advantages of the two-stage approach, including curtailing anti-competitive strategies and simplified bidding. But it does suggest that there may be very compelling reasons to use a mechanism other than VCG for the assignment stage—for example, a core-selecting mechanism. Indeed, the published rules for the FCC's upcoming 24 GHz auction, which largely follows the same rules as the forward part of the Broadcast Incentive Auction, switches from VCG to a core-selecting mechanism for the assignment stage.

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Online Appendix

Appendix A – Additional Examples

This appendix contains several examples that are referenced in the paper.

Example1: Existence of Multiple Truthful Subgame Perfect Equilibria

In this example, there are two blocks (A and B) and three bidders with single-unit demands. Both Bidder 1 and 2 prefer block A to block B while Bidder 3 is indifferent. It is easy to verify that there are two TSPEs in this setting, one with the efficient assignment and another with an inefficient assignment.

Table A1: Possibility of Multiple Truthful Subgame Perfect Equilibria

		Bidder 1	Bidder 2	Bidder 3
	Values	$v_1(A) = 110$	$v_2(A) = 109$	$v_3(A) = 102$
		$v_1(B) = 100$	$v_2(B) = 100$	$v_3(B) = 102$
	VCG Outcome	$(A,\ 109)$	$(\varnothing,\ 0)$	$(B,\ 100)$
		Multiple TSPEs of the Two-Stage Vickrey Auction		
TSPE 1 (efficient)	*Beliefs*	$\mu_1(1) = (1,0,1)$	$\mu_2(1) = (1,1,0)$	$\mu_3(1) = (1,0,1)$
	Bids in Stage 1	$b_1^I(1) = 110$	$b_2^I(1) = 100$	$b_3^I(1) = 102$
	Outcome of Stage 1	$(y_1 = 1,\ 100)$	$(y_2 = 0,\ 0)$	$(y_3 = 1,\ 100)$
	Bids in Stage 2	$b_1^{II}(A) = 10$		
	TSPE Outcome	$(A,\ 100)$	$(\varnothing,\ 0)$	$(B,\ 100)$
TSPE 2 (inefficient)	*Beliefs*	$\mu_1(1) = (1,1,0)$	$\mu_2(1) = (0,1,1)$	$\mu_3(1) = (0,1,1)$
	Bids in Stage 1	$b_1^I(1) = 101$	$b_2^I(1) = 109$	$b_3^I(1) = 102$
	Outcome of Stage 1	$(y_1 = 0,\ 0)$	$(y_2 = 1,\ 101)$	$(y_3 = 1,\ 101)$
	Bids in Stage 2		$b_2^{II}(A) = 9$	
	TSPE Outcome	$(\varnothing,\ 0)$	$(A,\ 101)$	$(B,\ 101)$

Note that the outcome of the two-stage Vickrey auction is outside the core with respect to true preferences, even when the outcome of the (one-stage) VCG mechanism is in the core. In the current example, a coalition that includes Bidder 2 and the auctioneer blocks the outcome of the efficient TSPE. Note that this outcome is obtained even though the outcomes of Vickrey auctions in both stages are in the core with respect to truthful bids.

Example 2: Non-existence of the Truthful Subgame Perfect Equilibrium

In this example, there are four blocks (A, B, C and D), three bidders and a contiguity restriction that restricts any multi-block winnings to form a contiguous segment. Bidder 1 only values two blocks (either AB, BC, or CD), and have assignment preference for winning BC (+20). Similarly, Bidder 2 only values two blocks (either AB, BC, or CD), and have assignment preference for winning AB (+10). Bidder 3 has a single-unit demand and have assignment preference for block A (+1) and block B (+15).

Table A2: Non-existence of Truthful Subgame Perfect Equilibrium

	Bidder 1	*Bidder 2*	*Bidder 3*
Values	$v_1(AB) = 100$	$v_2(AB) = 110$	$v_3(A) = 101$
	$v_1(BC) = 120$	$v_2(BC) = 100$	$v_3(B) = 115$
	$v_1(CD) = 100$	$v_2(CD) = 100$	$v_3(C) = 100$
			$v_3(D) = 100$
VCG Outcome	$(BC,\ 114)$	$(\varnothing,\ 0)$	$(A,\ 90)$

This example does not have any TSPEs by the following argument. In any TSPE, it must be the case that $b_1^I(2) \geq 100$, $b_2^I(2) \geq 100$ and $b_3^I(1) \geq 100$ resulting in exactly two winners of the allocation stage. Consider each scenario in turn:

1. Bidder 1 wins 2 blocks and Bidder 2 wins 2 blocks if $b_1^I(2) > b_3^I(1),\ b_2^I(2) > b_3^I(1)$. The corresponding beliefs are $\mu_1(2) = (2,2,0)$ and $\mu_2(2) = (2,2,0)$ resulting in $b_1^I(2) = 100$ and $b_2^I(2) = 110$. But then the belief of Bidder 3 must be $\mu_3(1) = (0,2,1)$ resulting in $b_3^I(1) = 105$, a contradiction.
2. Bidder 1 wins 2 blocks and Bidder 3 wins 1 block if $b_1^I(2) > b_2^I(2),\ b_3^I(1) > b_2^I(2)$. The corresponding beliefs are $\mu_1(2) = (2,0,1)$ and $\mu_3(1) = (2,0,1)$ resulting in $b_1^I(2) = 106$ and $b_3^I(1) = 101$. But then the belief of Bidder 2 must be $\mu_2(2) = (2,2,0)$ resulting in $b_2^I(2) = 110$, a contradiction.
3. Bidder 2 wins 2 blocks and Bidder 3 wins 1 block if $b_2^I(2) > b_1^I(2),\ b_3^I(1) > b_1^I(2)$. The corresponding beliefs are $\mu_2(2) = (0,2,1)$ and $\mu_3(1) = (0,2,1)$ resulting in $b_2^I(2) = 100$ and $b_3^I(1) = 105$. But then the belief of Bidder 1 must be $\mu_1(2) = (2,0,1)$ resulting in $b_1^I(2) = 106$, a contradiction.

Example 3: Non-core Outcome for Preferences satisfying Gross Substitution

In this example, there are two blocks (A and B) and each bidder's preferences satisfy gross substitutes: Bidder 1 has additive values over two blocks while Bidders 2 and 3 have single-unit demands. In the unique TSPE, the outcome of the first stage is outside the core with respect to bids (i.e., it is *visibly* outside the core): Bidders 2 and 3 jointly pay 10 for winning two generic blocks while Bidder 1 offered to pay 15. Thus, the well-behaved preferences and truthful bidding do not guarantee that a core outcome would emerge in the two-stage Vickrey auction.

Table A3: Example with GS Preferences

	Bidder 1	*Bidder 2*	*Bidder 3*
Values	$v_1(A) = 5,\ v_1(B) = 10$ $v_1(AB) = 15$	$v_2(A) = 15$ $v_2(B) = 15$	$v_3(A) = 16$ $v_3(B) = 22$
VCG Outcome	$(\varnothing,\ 0)$	$(A,\ 5)$	$(B,\ 10)$

Unique TSPE of the Two-Stage Vickrey Auction

Beliefs	$\mu_1(1) = (1,0,1)$ $\mu_1(2) = (2,0,0)$	$\mu_2(1) = (0,1,1)$	$\mu_3(1) = (0,1,1)$
Bids in Stage 1	$b_1^I(1) = 5,\ b_1^I(2) = 15$	$b_2^I(1) = 15$	$b_3^I(1) = 22$
Outcome of Stage 1	$(y_1 = 0,\ 0)$	$(y_2 = 1,\ 5)$	$(y_3 = 1,\ 5)$
Bids in Stage 2			$b_3^{II}(B) = 6$
TSPE Outcome	$(\varnothing,\ 0)$	$(A,\ 5)$	$(B,\ 5)$

Appendix B – Proofs

This appendix contains proofs.

Proof of Proposition 2

To prove existence, we need to show existence of correct beliefs. Due to assumption (A4), for any $y \in Y_i(q)$ and for the truthful bids under any beliefs $\{ \mu_i(\cdot) \}_{i=1}^n$, we have:

$$\sum_{\Delta(y,\tilde{y})} b_j^I(y_j) \leq \sum_{\Delta(y,\tilde{y})} \overline{v}_j(y_j) < \sum_{\Delta(y,\tilde{y})} \underline{v}_j(\tilde{y}_j) \leq \sum_{\Delta(y,\tilde{y})} b_j^I(\tilde{y}_j)$$

implying that allocation $\tilde{y} \in Y_i(q)$ is the only allocation in $Y_i(q)$ that can be consistent with the first-stage bids. Thus, setting the belief of bidder i for quantity q to the corresponding $\tilde{y}$ (i.e., $\mu_i(q) = \tilde{y}$) proves existence and uniqueness of TSPE. Efficiency of the equilibrium assignment follows immediately from the definition of TSPE and assumption (A5). *QED.*

Proof of Proposition 4

Preliminaries

The assignment stage consists of M blocks and n bidders with generic winnings $y^* = (y_1^*, ..., y_n^*)$ where $0 \leq y_i^* \leq m$ for each bidder i and $\sum_N y_j^* \leq m$ (if $y_i^* = 0$, then bidder i is not bidding in the assignment stage). For each bidder $i \in N$, define the assignment value function $\tilde{v}_i(.)$ as follows:

$$\tilde{v}_i(z) = \begin{cases} v_i(z) - \underline{v}_i(y_i^*) & z \in \Omega(y_i^*) \\ -\infty & z \notin \Omega(y_i^*) \end{cases}$$

Definition: For a given assignment stage with $y^* = (y_1^*, ..., y_n^*)$, the coalitional value function for bidders in coalition $C \subseteq N$ and a subset of blocks $z \subseteq M$ is given by:

$$w_C(z) = \max_{x \in X} \sum_C \tilde{v}_j(x_j),$$

where X is the set of all feasible assignments as defined in (3.1). By construction, $w_C(z) = -\infty$ for any subset of block z such that $|z| < \sum_C y_j^*$, $w_C(z) \geq 0$ for any z such that $|z| \geq \sum_C y_j^*$, and $w_C(\cdot)$ is monotone.

Definition: For a given coalition $C \subseteq N$, the coalitional value function $w_C(.)$ is *submodular in goods* if for any two subsets z and z' such that $z \subseteq z' \subseteq M$ and $|z| \geq \sum_C y_j^*$, and any subset of blocks k such that $k \cap z' = \varnothing$

$$w_C(z \cup k) \; - \; w_C(z) \; \geq \; w_C(z' \cup k) \; - \; w_C(z').$$

Definition: The coalitional value function is *submodular in bidders* if for any two coalitions C and C' such that $C \subseteq C' \subseteq N$, and any bidder $i \notin C'$

$$w_{C \cup i}(M) \; - \; w_C(M) \; \geq \; w_{C' \cup i}(M) \; - \; w_{C'}(M).$$

Definition: The coalitional value function is exhibiting *increasing marginal values* if for any coalition $C \subseteq N$, bidder $i \notin C$, and any two subsets z and z' such that $z \subseteq z' \subseteq M$ and $|z| \geq \sum_{C \cup i} y_j^*$

$$w_C(z') \; - \; w_C(z) \; \leq \; w_{C \cup i}(z') \; - \; w_{C \cup i}(z)$$

Supplementary Lemmas

Lemma 1. *If the coalitional value function is submodular in goods, then it exhibits increasing marginal values.*

Proof: Consider coalition $C \subseteq N$, bidder $i \notin C$, and any two subsets z and z' such that $z \subseteq z' \subseteq M$ and $|z| \geq \sum_{C \cup i} y_j^*$. Suppose that bidder i is assigned a subset of blocks k in the optimal assignment of blocks in z among members of coalition $C \cup i$, i.e., $w_{C \cup i}(z) \; = \; \tilde{v}_i(k) + w_C(z \setminus k)$. Since the same assignment for bidder i is feasible when assigning blocks in z' among members of coalition $C \cup i$, it must be the case that

$$w_{C \cup i}(z') \; \geq \; \tilde{v}_i(k) + w_C(z' \setminus k).$$

It follows that

$$\begin{aligned} w_{C \cup i}(z') \; - \; w_{C \cup i}(z) \; &\geq \; w_C(z' \setminus k) \; - \; w_C(z \setminus k) \\ &\geq \; w_C(z') \; - \; w_C(z) \end{aligned}$$

where the last inequality is due to submodularity in goods. *QED*.

Lemma 2. *If the coalitional value function exhibits increasing marginal values, then it is submodular in bidders.*

Proof: Consider two coalitions C and C' such that $C \subseteq C' \subseteq N$. First, note that increasing marginal values property implies that for any two subsets z and z' such that $z \subseteq z' \subseteq M$ and $|z| \geq \sum_{C'} y_j^*$, it must be the case that

$$w_C(z) \; - \; w_C(z') \; \geq \; w_{C'}(z) \; - \; w_{C'}(z')$$

Given bidder $i \notin C'$, suppose that bidder i is assigned a subset of blocks k in the optimal assignment of blocks in M among members of coalition $C' \cup i$, i.e., $w_{C' \cup i}(M) = \tilde{v}_i(k) + w_{C'}(M \setminus k)$. Since the same assignment for bidder i is feasible when assigning blocks in M among members of coalition $C \cup i$, it must be the case that

$$w_{C \cup i}(M) \geq \tilde{v}_i(k) + w_C(M \setminus k).$$

It follows that

$$\begin{aligned} w_{C \cup i}(M) - w_C(M) &\geq w_{C' \cup i}(M) - w_{C'}(M \setminus k) + w_C(M \setminus k) - w_C(M) \\ &\geq w_{C' \cup i}(M) - w_{C'}(M \setminus k) + w_{C'}(M \setminus k) - w_{C'}(M), \\ &= w_{C' \cup i}(M) - w_{C'}(M) \end{aligned}$$

where the second inequality is due to increasing marginal values. *QED*.

Proof of Proposition 4

For each bidder $i \in N$, assignment substitutes for $v_i(\cdot)$ for any $q = 1, ..., m$ trivially imply gross substitutes for the assignment value function $\tilde{v}_i(\cdot)$. Since the collection of assignment value functions $\{\tilde{v}_i(\cdot)\}_{i=1}^n$ satisfies gross substitutes, then, for any given coalition $C \subseteq N$, the coalitional value function $w_C(\cdot)$ also satisfies gross substitutes (Remark 6.1., Murota (2016)) and, consequently, submodularity in goods (pp.160, Murota (2016)). By Lemmas 1 and 2, the submodularity in goods for the coalitional value function implies submodularity in bidders. Finally, by Theorem 7 from Ausubel and Milgrom (2002), the Vickrey outcome in the assignment stage is in the core when the coalitional value function is submodular in bidders. *QED*.