An Efficient Ascending Auction for Private Valuations*

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Abstract

Dynamic auctions routinely elicit preferences by soliciting bidders' demands at posted prices. Due to this limitation, all known dynamic implementations of the Vickrey-Clarke-Groves (VCG) mechanism for general private valuations are impractical. In this paper, we design an ascending auction that uses linear and anonymous prices for elicitation and supports truthful bidding as an ex-post equilibrium. To the best of our knowledge, it is the first ascending auction that implements the VCG outcome for general environments while using practical elicitation methods.

Keywords: Combinatorial auctions, Multi-item auctions, Vickrey auctions, Dynamic Auctions, Ascending Auctions

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Auction design is tasked with developing market mechanisms for allocating scarce resources. In many applications, the primary goal for an auctioneer is *efficiency* — allocating resources to bidders with the highest intrinsic values. The famous Vickrey-Clarke-Groves (VCG) mechanism is essentially the unique mechanism that achieves efficiency in dominant strategies without requiring money transfers by losing bidders.[1] In addition, recent studies have shown that the VCG mechanism has some compelling properties in broader settings. For example, the VCG mechanism induces efficient ex-ante investments that predetermine bidders' values and can be a revenue-maximizing choice for auctioneers competing for bidders.[2]

Major efforts have been devoted to designing dynamic versions of the VCG mechanism.[3] Dynamic auctions are of interest due to the consensus that they are preferred over sealed-bid auctions in many applications. Benefits that are traditionally associated with dynamic auctions include avoiding disclosure of winning values (privacy preservation), avoiding the winner's curse through auction feedback, opportunities to manage budget constraints and reducing bidders' cognitive burden for placing bids.[4]

For simple settings, like selling or buying a single item, the main elements of a dynamic VCG implementation are well-understood (e.g., the English auction). For more general settings, the auction literature has repeatedly attempted to characterize the main attributes of efficient dynamic auctions. In fact, Gul and Stacchetti (2000) started their paper by putting forward the following definition of a dynamic auction:

> *"A dynamic auction can be described as a rule for adjusting prices given the observed history of demand (i.e., bids) and a rule for terminating the price adjustment procedure and specifying an allocation (i.e., determining who gets the good(s) and at what price(s)). The English auction is also identified with the property that prices are non-decreasing. More specifically, the English auction is typically identified with the procedure of increasing the prices as long as there is excess demand."*

The task of formally defining the class of dynamic auctions is hard since their main virtues (e.g., privacy preservation) are defined informally. Not surprisingly,

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[1] VCG is due to Vickrey (1961), Clarke (1971) and Groves (1973). For the uniqueness result, see Green and Laffont (1979) and Holmstrom (1979).

[2] See Hatfield, Kojima and Kominers (2017) and Jehiel and Lamy (2017).

[3] See Demange et al. (1986), Gul and Stacchetti (2000), Parkes and Ungar (2000 and 2002), Ausubel and Milgrom (2002), Bikhchandani and Ostroy (2002 and 2006), Ausubel (2004 and 2006), de Vries et al. (2007), Mishra and Parkes (2007) and Lamy (2012). A comprehensive survey of this literature is provided by Parkes (2006).

[4] Reduction in cognitive burden is captured by the notion of *obvious strategy-proofness* in Li (2017).

the definition from Gul and Stacchetti (2000) was found to be rather restrictive and was relaxed in subsequent work. For example, Ausubel (2006) allowed the auctioneer to collect bidders' demands along multiple price paths, while de Vries et al. (2007) let the auctioneer to solicit demands along a non-linear (i.e., non-additive over items) and non-anonymous (i.e., specific for every bidder) price path. While these relaxations are of theoretical interest, they are hardly practical since they involve rather awkward bidding procedures.

Despite many attempts to characterize dynamic auctions, one element of the original definition from Gul and Stacchetti (2000) has not been challenged — the auctioneer is restricted to quoting prices and collecting bidders' demands at these prices (i.e., the auctioneer is limited to use demand queries). In this paper, we investigate whether relaxing this restriction enables new practical auction designs.

The motivation for our inquiry comes from the field where several modern auction designs have already adopted flexible ways for eliciting bidders' preferences. The Combinatorial Clock Auction has been recently utilized for many spectrum auctions worldwide. This auction format uses standard demand queries during the initial phase of the auction, and it allows additional sealed bids in the last round to supplement the previously revealed information. In 2016, the Combinatorial Multi-Round Ascending Auction (CMRA) was used to allocate 1800 MHz spectrum in Denmark. The CMRA uses standard demand queries, but it also allows bidders to submit alternative bids in each round. While not implemented in practice at the time of this writing, Baranov et al. (2017) develop a new elicitation procedure suitable for settings with increasing returns. Under their proposal, an auctioneer quotes a price and bidders list all quantities they are willing to buy instead of listing only their preferred quantities.

For the general setting with private values, an ascending auction that implements the VCG outcome was first proposed by Mishra and Parkes (2007). Their design uses demand queries and requires a non-linear and non-anonymous price path, a limitation that handicaps the design's practical appeal. In this paper, we adopt several insights from Mishra and Parkes (2007) as a starting point.

Our first contribution is a general class of iterative Vickrey auctions. We refer to an auction as iterative if it uses an iterative process to elicit bidders' preferences (i.e., bidders reveal their preferences in a step-by-step manner).[5] The class of iterative Vickrey auctions generalizes the class of ascending auctions introduced by Mishra and Parkes (2007) since it does not limit the auctioneer to a particular

[5]We intentionally distinguish between iterative auctions and dynamic auctions to avoid dealing with the problem of formally defining dynamic auctions. A dynamic auction has to be iterative since bidders must have multiple opportunities to communicate their preferences, but iterative auctions that do not have any advantages associated with dynamic auctions should not be referred to as dynamic.

elicitation process. Instead, our class admits any elicitation process as long as it satisfies two natural conditions. We prove that any iterative auction in this class implements the VCG outcome as an ex-post equilibrium.[6] The result has important implications for designing auctions — to design an efficient auction for a nonstandard setting, an auctioneer can develop a customized elicitation process that fits the application in hand and use standard procedures for all other components (e.g., a closing rule) to complete its design.[7]

Our main contribution is a characterization of an efficient ascending auction for general private valuations that implements the VCG outcome as an ex-post equilibrium while using a single linear and anonymous price path for elicitation. The insufficiency of linear and anonymous prices is circumvented by allowing the auctioneer to make additional inquiries in situations where standard demand queries would miss critical information.[8]

We illustrate our approach with a simple example. Consider a homogeneous setting with three identical items and a bidder with the following values $v(1) = 20$, $v(2) = 25$, $v(3) = 40$. Suppose that the auctioneer quotes a per unit price p and asks the bidder to report its demand. Then the auctioneer cannot elicit bidder's value for 2 units since bidder's demand is 3 when $p \leq 10$ and 1 when $p > 10$; and the VCG outcome cannot be implemented in case $v(2)$ is required. Now suppose that the auctioneer can ask bidders to report their marginal values for any missed quantities at any time their demand drops by more than one unit. Then, the auctioneer would elicit $v(2)$ when the bidder drops its demand from 3 to 1 at $p = 10$. We show that this simple addition to the auctioneer's elicitation capabilities suffices to resolve the insufficiency of linear prices and produce a dynamic VCG implementation for the general setting that is governed by a simple price path.

The article is organized as follows. Section 1 describes the model. The general framework for iterative Vickrey auctions is presented in Section 2, and the new ascending auction is described in Section 3. Section 4 discusses several implementation issues and Section 5 concludes. The appendix contains all proofs and technical details.

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[6]This is a common result that appears in Gul and Stacchetti (2000), Ausubel (2004 and 2006), Bikhchandani and Ostroy (2006), de Vries et al. (2007) and Mishra and Parkes (2007) for their respective settings and auction designs.

[7]For example, Baranov et al. (2017) introduce a new "interval bidding" elicitation technique and use it as a main component to design an efficient auction for a procurement setting with increasing returns.

[8]For the insufficiency of linear prices, see Gul and Stacchetti (2000), Bikhchandani and Ostroy (2002), and Mishra and Parkes (2007).

1 Model

A seller offers multiple units of K heterogeneous indivisible goods, denoted by vector $S = \{s^1, ..., s^K\} \in Z^K_{++}$, to a set of bidders $N = \{1, ..., n\}$. The set of all possible bundles of items is denoted by $\Omega = \{(z^1, ..., z^K) : 0 \leq z^k \leq s^k \quad \forall k \in \{1, ..., K\}\}$. For every bidder $i \in N$, and every bundle $z \in \Omega$, the valuation of bidder i is given by $v_i(z)$, and the bidder's value for the null bundle is normalized to zero. We make the following standard assumptions:

(A1) *Pure Private Values:* Each bidder i knows its own valuation for any bundle z, and this valuation does not depend on valuations of other bidders;

(A2) *Quasilinear Values:* The payoff of bidder i from winning bundle z in exchange for a payment y is given by $v_i(z) - y$;

Another standard assumption made in the literature is monotonicity of value functions (i.e., free disposal). This assumption is not needed for our results.

An allocation $x = (x_1, ..., x_n)$ is called *feasible* if $x_i \in \Omega$ for all $i \in N$ and $\sum_N x_j \leq S$. The set of all feasible allocations is denoted by X. We denote $E(M)$ an economy that only includes bidders in $M \subseteq N$.

The *coalitional value function* for bidders in coalition $M \subseteq N$ is given by:

$$w(M) \ = \ \max_{x \in X} \ \textstyle\sum_M v_j(x_j). \tag{1.1}$$

A feasible allocation $x = (x_1, ..., x_n) \in X$ is *efficient* for economy $E(M)$ if

$$\textstyle\sum_M v_j(x_j) \ = \ w(M). \tag{1.2}$$

A *Vickrey outcome* consists of an efficient allocation vector $x^* = (x^*_1, ..., x^*_n)$ for the main economy and a corresponding payment vector $y^V = (y^V_1, ..., y^V_n)$ where $y^V_i = w(N_{-i}) - \sum_{N_{-i}} v_j(x^*_j)$ for all $i \in N$ where N_{-i} denotes the coalition of all bidders in N excluding bidder i.

2 Iterative Vickrey Auctions

In this section, we describe a general class of iterative auctions that implement the Vickrey outcome.

2.1 Preliminaries

An auction must elicit sufficient information about bidders' value functions to confirm the efficiency of a particular allocation. The integral part of this task is a process for preference elicitation. Most elicitation processes are designed after the famous *"Walrasian auctioneer"* by Walras (1874) — the auctioneer quotes prices and asks bidders to report their demand at these prices (i.e., demand queries). However, an elicitation protocol based only on demand queries can be too restrictive and the auctioneer might benefit from using other types of queries.

A general framework for an iterative auction can be described as follows. At each time $t \geq 0$ (where $t = 0$ is the starting time), the auctioneer asks bidders to provide some information about their preferences and for each bidder i constructs an approximation of its value function $\hat{v}_i(., t)$ using bidder i's responses received on the time interval $[0, t]$. Here we assume that the auctioneer always generates a unique $\hat{v}_i(z, t)$ for each bundle $z \in \Omega$, and that $\hat{v}_i(\emptyset, t) = 0$ for any $t \geq 0$.

For each bidder i, denote the approximation error for bundle z at time t as

$$\delta_i(z, t) \;=\; v_i(z) \;-\; \hat{v}_i(z, t), \tag{2.1}$$

and let $\Delta_i(t)$ represent a set that contains all bundles with the highest approximation error at time t, i.e.:

$$\Delta_i(t) \;=\; \arg\max_{z \in \Omega} \delta_i(z, t). \tag{2.2}$$

By construction, $\delta_i(z, t) \geq 0$ for any bundle $z \in \Delta_i(t)$ since the null bundle belongs to Ω. Also note that set $\Delta_i(t)$ contains bundles for which true marginal values relative to each other can be recovered using $\hat{v}_i(., t)$.

Using current approximations of value functions, the auctioneer finds $x^*(M, t)$, a tentative value-maximizing allocation for economy $E(M)$ at time t, by solving the following winner determination problem:

$$x^*(M, t) \;\in\; X^*(M, t) \;=\; \arg\max_{x \in X} \textstyle\sum_M \hat{v}_j(x_j, t) \tag{2.3}$$

When allocation $x^*(M, t)$ is not unique, the auctioneer picks one of the allocations in $X^*(M, t)$ using some criteria.

Proposition 1 below is a partial restatement of the result obtained by Parkes (2002) and Nisan and Segal (2006) using our notation.[9] It provides a sufficient condition to prove the efficiency of a tentative allocation $x^*(M, t)$ for economy $E(M)$.

Proposition 1. *[Parkes (2002), Nisan and Segal (2006)] A tentative allocation* $x^*(M, t) = (x_1, ..., x_n)$ *is efficient for economy* $E(M)$ *if*

$$x_i \;\in\; \Delta_i(t) \qquad \forall i \in M. \tag{2.4}$$

[9] Parkes (2002) and Nisan and Segal (2006) provide the necessary and sufficient conditions. In our case, only the sufficiency part can be established. Note that Proposition 1 corresponds to the First Fundamental Welfare Theorem with non-linear and non-anonymous prices.

Next proposition, due to Parkes and Ungar (2002) and Lahaie and Parkes (2004), provides the necessary and sufficient condition for identifying the Vickrey outcome.[10]

Proposition 2. *[Parkes and Ungar (2002), Lahaie and Parkes (2004)] Suppose that at time t, condition (2.4) is satisfied for the main economy $E(N)$. Then the Vickrey outcome can be identified from $\{\hat{v}_j(., t)\}_{j \in N}$ if and only if condition (2.4) is also satisfied for all marginal economies $\{E(N_{-1}), E(N_{-2}), ..., E(N_{-n}\})$ at time t.*

Propositions 1 and 2 highlight an important role of condition (2.4). The only way to implement the Vickrey outcome using approximations of value functions $\{\hat{v}_j(., t)\}_{j \in N}$ instead of true value functions is to uncover enough information to satisfy condition (2.4) in all relevant economies. However, this condition cannot be verified directly since set $\Delta_i(t)$ depends on $v_i(.)$. Therefore, an elicitation process must construct not only an approximation of the value function $\hat{v}_i(., t)$ but also a replacement for set $\Delta_i(t)$ that can be used to verify this condition.

2.2 Elicitation Process

In this section, we formally define elicitation processes and propose several properties that are useful for constructing iterative auctions. An elicitation process is called *fully expressive* if types of queries that the auctioneer can make are sufficient to elicit any value function that satisfies assumptions (A1) - (A2) from a bidder who responds truthfully.[11] For the rest of the paper, we consider only fully expressive elicitation processes.

First, we formally define an elicitation process. Our definition highlights the equally important role of two main components: an approximation of the value function $\hat{v}_i(., t)$ and a feasible replacement for set $\Delta_i(t)$, denoted by $\widehat{\Delta}_i(t)$.

Definition 1. *An **elicitation process** is a procedure that for each time $t \geq 0$ specifies informational queries addressed to each bidder i and converts bidder i's responses received on $[0, t]$ into a single-valued function $\hat{v}_i(., t)$ and a set of bundles $\widehat{\Delta}_i(t)$.*

This definition is less innocent than it sounds. It implies that an elicitation process is sophisticated enough to resolve any inconsistencies in bidder i's responses to always produce a single-valued function $\hat{v}_i(., t)$ and set $\widehat{\Delta}_i(t)$.

The first property that we propose links an elicitation process and the true value function — all responses received from a truthful bidder should be treated

[10] This result also appears as Theorem 1 in Mishra and Parkes (2007).

[11] The term "fully expressive" comes from the literature on bidding languages (see Nisan (2006)).

as such; otherwise, a bidder wishing to communicate its true values would have to respond untruthfully.

Definition 2. *An elicitation process is called* ***straightforward*** *if it converts truthful responses of each bidder i made on the interval $[0, t]$ into a function $\hat{v}_i(., t)$ and set $\widehat{\Delta}_i(t)$ such that:*

$$\widehat{\Delta}_i(t) \;\subseteq\; \Delta_i(t), \tag{2.5}$$

where $\Delta_i(t)$ is defined by (2.2) for $\hat{v}_i(., t)$ and true $v_i(.)$.

This property is a key for testing condition (2.4) as it allows the auctioneer to use $\widehat{\Delta}_i(t)$ instead of $\Delta_i(t)$.

The second property that we propose requires that an elicitation process is capable, if needed, to fully elicit bidder i's value function, and the complete elicitation implies that $\Delta_i(t) = \Omega$. To guarantee this property, we require that an elicitation process weakly expands set $\widehat{\Delta}_i(t)$ over time. In other words, the auctioneer progressively builds up the set of bundles that can be awarded to bidder i while never discarding bundles that have been already added to $\widehat{\Delta}_i(t)$.

Definition 3. *An elicitation process is called* ***iterative*** *if the set $\widehat{\Delta}_i(t)$ for each bidder i is weakly increasing in time, i.e.*

$$\widehat{\Delta}_i(t') \;\subseteq\; \widehat{\Delta}_i(t) \qquad \textit{for all} \quad t' \leq t. \tag{2.6}$$

Next, we define several monotonicity restrictions that can be imposed on an elicitation process. These restrictions are not required for our main results, but they are frequently used in practice to yield a monotonic discovery of an auction outcome. One of the most popular restrictions utilized in practice is a restriction on a direction for adjustments to approximations of value functions. These restrictions ensure that the approximation monotonically approaches the true value function from below or from above.

Definition 4. *An iterative elicitation process is called* ***ascending (descending)*** *if for each bidder i and any bundle $z \in \widehat{\Delta}_i(t)$ ($z \notin \widehat{\Delta}_i(t)$), $\hat{v}_i(z, t)$ is non-decreasing (non-increasing) in time.*[12]

Another desirable form of monotonicity in iterative auctions is a guarantee that once a tentative allocation has been proven optimal for economy $E(M)$, it continues to be optimal in the future. We say that economy $E(M)$ is *cleared at time t* if there exists a tentative allocation $x^*(M, t) = (x_1, ..., x_n)$ that solves (2.3) such that

[12]This definition is weaker than the one provided by Mishra and Parkes (2007) where the direction for adjustments is constrained for all bundles $z \in \Omega$.

$$x_i \ \in \ \widehat{\Delta}_i(t) \qquad \forall i \in M. \tag{2.7}$$

It is easy to verify that the next property guarantees that a cleared economy stays cleared at a later time by ensuring that value adjustments for all bundles in $\widehat{\Delta}_i(t)$ are at least as high as for bundles outside of $\widehat{\Delta}_i(t)$.

Definition 5. *An iterative elicitation process is **monotonic** if for each bidder i, any time $t' \leq t$ and any pair of bundles $z \in \Omega$ and $y \in \widehat{\Delta}_i(t')$:*

$$\hat{v}_i(z,t) \ - \ \hat{v}_i(z,t') \ \leq \ \hat{v}_i(y,t) \ - \ \hat{v}_i(y,t'). \tag{2.8}$$

2.3 Iterative Vickrey Auctions

In this section, we define a general class of iterative Vickrey auctions and show that they implement the Vickrey outcome as an ex -post equilibrium. First, we specify an adjustment rule that moves the elicitation process along by forcing bidders to successively reveal more competitive values. To facilitate greater generality, we only require that at any time at least one bidder is asked to revise its values to guarantee the termination of the process in a finite time.

Definition 6. *An iterative elicitation process satisfies the adjustment rule if there exist $\epsilon > 0$ and $\lambda > 0$ such that for any $t \ \geq \ t' + \lambda$ there exists bidder $i \in N$ for whom either*

$$\hat{v}_i(z,t) \ - \ \hat{v}_i(z,t') \ \geq \ \epsilon \qquad \textit{for all} \qquad z \ \in \ \widehat{\Delta}_i(t'), \quad \textit{or} \tag{2.9}$$

$$\hat{v}_i(z,t') \ - \ \hat{v}_i(z,t) \ \geq \ \epsilon \qquad \textit{for some} \quad z \ \notin \ \widehat{\Delta}_i(t'). \tag{2.10}$$

Intuitively, condition (2.9) forces bidder i to increase values for all bundles in $\widehat{\Delta}_i(t')$, an adjustment typical for an ascending auction in which bidders increase the implicit values for their current demands (like in the English auction). Alternatively, condition (2.10) forces bidder i to decrease values for some bundles outside of $\widehat{\Delta}_i(t')$, an adjustment typical for a descending auction where the implicit values for bundles decrease until bidders start demanding them (like in the Dutch auction). Now we define a general class of iterative Vickrey auctions.

Definition 7. *An iterative Vickrey auction is an auction procedure that:*

(1) uses a straightforward and iterative elicitation process that satisfies the adjustment rule;

(2) terminates when all economies in $\{E(N), E(N_{-1}), ..., E(N_{-n})\}$ are cleared at the same time (T denotes the termination time);

(3) awards bundle $x_i^*(N,T)$ to bidder i in exchange for a payment

$$y_i^V \ = \ \sum_{j \in N_{-i}} \left[\hat{v}_j(x_j^*(N_{-i},T)) \ - \ \hat{v}_j(x_j^*(N,T)) \right]. \tag{2.11}$$

Our class of iterative Vickrey auctions is intentionally very permissive due to a flexible adjustment rule. For example, the class permits implementations that should be considered sealed-bid rather than dynamic.[13] In practice, the auctioneer should design an elicitation process and an adjustment rule in a way that delivers advantages of dynamic auctions.

Theorem 1. *If each bidder i bids truthfully according to $v_i(.)$, an iterative Vickrey auction implements the Vickrey outcome, and truthful bidding by all bidders is an ex-post equilibrium.*

Theorem 1 generalizes the standard result in the literature that was previously established for specific elicitation processes and specific settings. More generally, it allows us to simplify the problem of designing a dynamic Vickrey auction to a task of developing a straightforward and iterative elicitation process. Auction designers can focus on creating elicitation processes that are tailored to their specific objectives such as speed, privacy, feedback, bidding convenience, and etc. In the next section, we adopt this approach to develop a dynamic Vickrey auction for the general private value setting that uses a single linear and anonymous ascending price path to guide preference elicitation.

3 Efficient Ascending Auction

This section contains our main result. We propose an efficient ascending auction with an elicitation process that is driven by a single linear and anonymous price path.

3.1 Elicitation Process

The auctioneer initializes K clock prices, one for each good, at zero. At any time $t \geq 0$, the auctioneer announces current clock prices $p(t)$, and each bidder i replies with a single bundle $x_i(t) \in \Omega$ that is treated as bidder i's demand at current prices. The price trajectory $p(.)$ is assumed to be non-decreasing, continuous and piecewise linear on $[0, +\infty)$. The demand function $x_i(.)$ for each bidder i is assumed to be a right-continuous piecewise constant function.[14]

[13]Consider an iterative auction where an auctioneer uncovers value functions sequentially (bidder 1, then bidder 2, then bidder 3, etc.) without providing feedback to bidders. This iterative auction should not be classified as dynamic since it does not have any properties associated with dynamic auctions.

[14]This assumption is without loss of generality. It can be shown that a piecewise constant demand function exists for any function $v(.)$ satisfying assumptions (A1) - (A2) provided that the price path $p(.)$ is continuous and piecewise linear on $[0, +\infty)$.

It is well known that demand queries in combination with a linear and anonymous price path do not produce a fully expressive elicitation process, so the auctioneer has to elicit more information when the need arise. A truthful bidder would never demand a bundle that is a superset of its demand at an earlier time when clock prices are non-decreasing ("law of demand"). Since values for such bundles cannot be elicited with any feasible demand queries, they must be recovered in some other way. One approach is to ask bidders to report their marginal values for such bundles relative to their current demand.

Denote $\widehat{\Delta}_i(t)$ the set of revealed bundles that includes any bundle $z \in \Omega$ that is a superset of bidder i's demand for some time $t' \in [0,t]$, i.e.:

$$\widehat{\Delta}_i(t) \;=\; \big\{ z \in \Omega : \; \exists\, t' \in [0,t] \;\; \text{such that} \;\; z \geq x_i(t') \,\big\}. \tag{3.1}$$

By construction, set of revealed bundles $\widehat{\Delta}_i(t)$ only expands when bidder i demands a new bundle at time t. At this time, bidder i is required to report its marginal values relative to its demand $x_i(t)$ for all bundles that are newly added to the set $\widehat{\Delta}_i(t)$. For any newly added bundle z, denote $\widetilde{mv}_i(z)$ the reported marginal value of bundle z relative to $x_i(t)$; and denote $t(z)$ the time when bundle z is added to $\widehat{\Delta}_i(t)$. Note that the number of bundles that are added to set $\widehat{\Delta}_i(t)$ at any given time can be large.

We say that bidder i *bids truthfully according to its value function* $v_i(.)$ on $[0,t]$ if at any time $s \in [0,t]$:

(a) bidder i truthfully reports its demand $x_i(s)$ given $p(s)$, i.e.,

$$x_i(s) \;\in\; \arg\max_{z \in \Omega} \big[v_i(z) \;-\; p(s)\, z \big]; \;\; \text{and} \tag{3.2}$$

(b) bidder i truthfully reports its marginal value for all bundles that are added to the set of revealed bundles at time s, i.e., for any bundle z that is added to $\widehat{\Delta}_i(s)$ at time s,

$$\widetilde{mv}_i(z) \;=\; v_i(z) \;-\; v_i(x_i(s)). \tag{3.3}$$

To ensure that each bidder bids according to some value function, our elicitation process requires activity rules that are based on *the Generalized Axiom of Revealed Preference (GARP).* The most famous result in the GARP literature is the *Afriat's Theorem* due to Afriat (1967). It establishes a direct connection between GARP and existence of a value function that rationalizes demand $x_i(.)$ given price path $p(.)$. Using our notation, the Afriat's theorem is stated as follows:

Afriat's Theorem (1967). *Given price path $p(.)$, bidder i's demand $x_i(.)$ is rationalized by a value function satisfying assumptions (A1) and (A2) if and only if its demand $x_i(.)$ satisfies GARP on $[0,t]$, i.e.:*

$$(GARP) \qquad p(s)[x_i(s) - x_i(s')] + \int_s^{s'} p(u)dx_i(u) \leq 0 \qquad \forall\, s, s' \in [0,t].$$

Ausubel and Baranov (2018) introduced the following notion of GARP violation. Suppose that bidder i's demand satisfies GARP on interval $[0,t]$. Then we can hypothetically ask whether bidder i would have violated GARP by demanding bundle z at time t by calculating $gv_i(z,t)$ defined as follows:

$$gv_i(t,z) = \max_{s\in[0,t]} \left\{ p(s)[x_i(s) - z] + \int_s^t p(u)dx_i(u) + p(t)[z - x_i(t)] \right\} \quad (3.4)$$

Intuitively, $gv_i(t,z)$ is the maximum net amount that can be extracted from bidder i in a series of transactions that starts and ends with the same bundle z. By construction, $gv_i(t,z) \geq 0$ for all $z \in \Omega$ and $gv_i(t,z) > 0$ indicates that bidding for bundle z at time t violates rationality (i.e., proves existence of a "money pump").[15] Furthermore, when bidder i bids truthfully according to $v_i(.)$, the upper bound on the marginal value of bundle z relative to $x_i(t)$ is given by:

$$v_i(z) - v_i(x_i(t)) \leq p(t)[z - x_i(t)] - gv_i(t,z). \quad (3.5)$$

For our purposes, we need two activity rules: one to restrict demand changes; and another one to limit the reported marginal value for bundles that are added to the set of revealed bundles.

AR1: At time t, bundle $z \in \Omega$ is unacceptable as demand of bidder i if either $gv_i(t,z) > 0$ or $z \geq x_i(s)$ for some time $s < t$;[16]

AR2: For bundle z that is added to $\widehat{\Delta}_i(t)$ at time t, the reported marginal value $\widetilde{mv}_i(z)$ has to satisfy the following inequality:

$$\widetilde{mv}_i(z) \leq p(t)[z - x_i(t)] - gv_i(t,z). \quad (3.6)$$

AR1 forces bidder i to submit demands that can be rationalized by some value function and also precludes bids for any supersets of bundles that were demanded before. AR2 restricts the marginal value report by the highest possible value for bundle z that is consistent with the fact that bidder i has never demanded bundle

[15] For a piecewise linear price path $p(.)$ and piecewise constant demand $x_i(.)$, the optimization problem (3.4) has a linear programming formulation that can be used to calculate $gv_i(t,z)$ in applications.

[16] When validating AR1, $gv_i(t,z)$ is calculated using (3.4) by setting $x_i(t) := z$.

z.[17] Proposition 3 proves that each bidder is forced to bid according to some value function when restricted by these activity rules.[18]

Proposition 3. *Given non-decreasing price path $p(.)$, bidder i bids truthfully according to some value function on $[0,t]$ if and only if its bidding is constrained by activity rules AR1 and AR2.*

Next, we construct the approximation of value function $\hat{v}_i(.,t)$ to complement the set of revealed bundles $\widehat{\Delta}_i(t)$. For any bundle $z \in \widehat{\Delta}_i(t)$, denote its revealed marginal value relative to bidder i's current demand $x_i(t)$ as:

$$mv_i(z,t) \;=\; \begin{cases} \int_t^{t'} p(u)dx_i(u) & \text{if } \; \exists\, t' \in [0,t] : \; x_i(t') = z \\ \int_t^{t(z)} p(u)dx_i(u) \;+\; \widetilde{mv}_i(z) & \text{otherwise} \end{cases} \tag{3.7}$$

Intuitively, the marginal value for bundle z is elicited either via revealed preferences if bundle z was previously demanded by bidder i, or its marginal value was elicited with an additional query at time $t(z)$ when the bundle was added to $\widehat{\Delta}_i(t)$. Using revealed marginal values, the auctioneer constructs the current approximation of the value function as follows:

$$\hat{v}_i(z,t) \;=\; \begin{cases} p(t)\, x_i(t) \;+\; mv_i(z,t) & \text{if} \quad z \in \widehat{\Delta}_i(t) \\ p(t)\, z \;-\; gv_i(t,z) & \text{if} \quad z \notin \widehat{\Delta}_i(t) \end{cases} \tag{3.8}$$

For any revealed bundle $z \in \widehat{\Delta}_i(t)$, formula (3.8) uses the revealed marginal value between bundle z and the current demand $x_i(t)$ imputing the current clock price for the current demand $x_i(t)$. For any non-revealed bundle $z \notin \widehat{\Delta}_i(t)$, the formula imputes the maximum possible value for bundle z that is consistent with the current bidding history of bidder i to ensure that these bundles create maximum competition for bundles in $\widehat{\Delta}_i(t)$. The latter part guarantees that the elicitation process is straightforward and monotonic.[19]

All properties of the elicitation process are summarized by Proposition 4.

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[17]For the homogeneous setting $K = 1$, AR1 simplifies to requiring $x_i(t) \le x_i(s)$ for any $s \in [0,t]$ (i.e., a non-increasing demand) and AR2 simplifies to $\widetilde{mv}_i(z) \;\le\; p(t)\,[z - x_i(t)]$ for any z that is added to $\widehat{\Delta}_i(t)$ at time t.

[18]Most interestingly, constraining reported marginal values with AR2 at time t turns out to be sufficient for existence of a value function that rationalizes bidding at later times. To put it differently, the upper bound for $\widetilde{mv}_i(z)$ from (3.6) derived at time t does not change later as long as prices are non-decreasing.

[19]If the auctioneer imputes $p(t)z$ instead of $p(t)z - gv_i(t,z)$, the elicitation process is straightforward but not monotonic.

Proposition 4. *If bidder i is constrained by activity rules AR1 and AR2, then $\hat{v}_i(.,t)$ as defined in (3.8) and $\widehat{\Delta}_i(t)$ as defined in (3.1) constitute an elicitation process that is straightforward, iterative, ascending and monotonic.*

Clock auctions traditionally use the excess demand to adjust clock prices and determine whether the auction reached its end. In the next section, we define an appropriate notion of excess demand for our auction and develop a clock price adjustment process.

3.2 Excess Demand and Clock Increments

The last ingredient for our ascending auction is a rule to adjust clock prices. Such rule has to be linked to the closing rule to guarantee that prices are incremented in a meaningful way until the closing rule is met. We adopt a standard approach of specifying a notion of excess demand with a property that the closing rule is satisfied if and only if there is no excess demand for any goods. When the closing rule is not satisfied, the excess demand is positive for at least one good and increasing the clock prices for goods with excess demand puts pressure on bidders to reveal more competitive values.[20]

By definition, economy $E(M)$ is cleared at time t if there exists a tentative assignment $x^*(M,t)$ that assigns each bidder in M a bundle from its corresponding set of revealed bundles $\widehat{\Delta}_i(t)$. The failure to clear economy $E(M)$ is traced to bidders whose tentative winnings are not in their sets of revealed bundles and motivates the following construction of the excess demand. When bidder i is assigned a bundle from $\widehat{\Delta}_i(t)$, bidder i does not prevent economy $E(M)$ from clearing, and its contribution towards the excess demand is zero. When bidder i is assigned a bundle outside of $\widehat{\Delta}_i(t)$, bidder i does prevent economy $E(M)$ from clearing, and its contribution towards the excess demand must account for its current demand for items that are not awarded to the bidder in its tentative assignment.

This construction has two sources of multiplicity. First, there can be multiple tentative value-maximizing allocations for economy $E(M)$, and second, some bidders might be implicitly demanding multiple bundles at $p(t)$. Define $D_i(t)$ the revealed demand correspondence of bidder i at time t as

$$D_i(t) \;=\; \operatorname*{arg\,max}_{z \in \widehat{\Delta}_i(t)} \left[\, \hat{v}_i(z,t) - p(t)\, z \,\right]. \tag{3.9}$$

[20]The classic notion of excess demand does not work in this environment since: (1) it does not match our closing rule that has to account for marginal economies; and (2) there is a possibility that an efficient allocation does not allocate all goods to bidders.

The set of excess demands *(excess correspondence)* for economy $E(M)$ at time t is defined as

$$Z(M,t) \;=\; \{\; \textstyle\sum_M z_j \;\} \tag{3.10}$$

such that there exists allocation $x^* \in X^*(M,t)$ and for each bidder $i \in M$, $z_i = 0$ if $x_i^* \in \widehat{\Delta}_i(t)$ and $z_i = \max\{\,0, d - x_i^*\,\}$ if $x_i^* \notin \widehat{\Delta}_i(t)$ where $d \in D_i(t)$.

The next proposition shows that the clearing of economy $E(M)$ at time t is equivalent to a natural "zero excess demand" condition.

Proposition 5. *If all bidders are constrained by activity rules AR1 and AR2, economy $E(M)$ is cleared at time t if and only if $0 \in Z(M,t)$.*

When economy $E(M)$ is not cleared at time t, any excess demand vector $z \in Z(M,t)$ is positive for at least one good. The auctioneer can direct the auction process towards clearing economy $E(M)$ by picking one of the excess demand vectors from $Z(M,t)$ and increasing clock prices for goods with excess demand. While there are many possible ways to pick one of the excess demand vectors from $Z(M,t)$, the simplest approach is to choose one that minimizes the current price value of excess demand, i.e.:

$$z(M,t) \;\in\; \underset{z \,\in\, Z(M,t)}{\arg\min}\; p(t)\, z \tag{3.11}$$

This approach offers an advantage of targeting the smallest excess demand when measured by the current clock prices.

To identify the Vickrey outcome, the auctioneer needs to clear the main economy and all marginal economies. An important consideration for the auctioneer is an order in which these economies are targeted. For ascending auctions, it is natural for marginal economies to clear ahead of the main economy due to weaker competition. Clearing marginal economies before the main economy is a desirable property since a different order can cause incentive problems in applications. For simplicity, we assume that the auctioneer simultaneously targets all relevant economies. We provide a more detailed discussion of this issue in Section 4.2.

For each product $k \in \{1, ..., K\}$, define a cumulative excess demand at time t as the maximum of excess demands for this product among all relevant economies $M \in \{N, N_{-1}, ..., N_{-n}\}$, i.e.,

$$Z^k(t) \;=\; \max_{M \in \{N, N_{-1}, \ldots, N_{-n}\}} z^k(M,t) \tag{3.12}$$

A naive price adjustment process based on the excess demand (which in turn depends on individual demands) can cause a known technical problem with infinitive price oscillations (see Gul and Stacchetti (2000) and Ausubel (2006)). To avoid this problem and to provide a way to implement this design in practice, we adopt a price adjustment process that regulates the speed of price clocks at discrete times. This approach is known as *intra-round bidding* and to the best of our knowledge, is the only practical way of implementing continuous price clocks. Formally, the auctioneer initializes price clocks at zero $p(0) = 0$ and asks for initial reports to construct $\widehat{\Delta}_i(0)$ for each bidder $i \in N$. Then, at any time $t \geq 0$, the auctioneer sets the clock price for good $k \in \{1, ..., K\}$ using the following formula:

$$p^k(t) \;=\; \begin{cases} p^k(t')\,[1 \;+\; \epsilon\,(t-t')] & \text{if} \quad Z^k(t') > 0 \\ p^k(t') & \text{if} \quad Z^k(t') = 0 \end{cases}, \tag{3.13}$$

where t' is the highest integer such that $t' < t$ and $\epsilon > 0$. Intuitively, the price adjustment in (3.13) uses the excess demand at integer time t' to determine which clock prices have to be increased on the time interval $(t', t'+1]$ and then increases them at a constant speed ϵ (for example, $\epsilon = 0.05$ corresponds to a 5% increment on the time interval $[t', t'+1]$ for goods with positive excess demand). This price process produces a non-decreasing continuous piecewise linear price path.

3.3 Efficient Ascending Auction

In this section, we specify our ascending auction and prove our main result (Theorem 2) that shows that it implements the Vickrey outcome as an ex-post equilibrium.

Ascending Auction: The ascending auction consists of the following components:

(1) The auctioneer initializes clock prices at zero $p(0) = 0$. At each time $t \geq 0$, the auctioneer quotes clock prices $p(t)$ and asks each bidder i to report (1) its demand $x_i(t)$; and (2) its marginal value for any bundle that is added to the set of revealed bundles $\widehat{\Delta}_i(t)$ at time t. All responses are subject to activity rules AR1 and AR2;

(2) At each time $t \geq 0$ and for each bidder i, the auctioneer constructs $\widehat{\Delta}_i(t)$ and $\hat{v}_i(, t)$ using formulas (3.1) and (3.8). In addition, the auctioneer calculates excess demand $Z(t)$ according to formulas (3.9) - (3.12);

(3) If $Z(t) \neq 0$ at time t, the clock prices are adjusted using the price adjustment process (3.13) and the process goes back to step (2). If $Z(t) = 0$, then the process terminates ($T := t$) and bidder i is awarded bundle $x_i^*(N, T)$ in exchange for a payment

$$y_i^V = \sum_{j \in N_{-i}} \left[\hat{v}_j(x_j^*(N_{-i}, T)) - \hat{v}_j(x_j^*(N, T))\right]. \tag{3.14}$$

Theorem 2. *If each bidder i bids truthfully according to $v_i(.)$, the ascending auction implements the Vickrey outcome, and truthful bidding by all bidders is an ex-post equilibrium.*

3.4 An Illustrative Example

To illustrate our ascending auction, we use an example with two items, A and B, and three bidders. Bidder 1 values item A at 3 and has no value for item B. Bidder 2 values item B at 10 and has no value for item A. Bidder 3 values the package AB at 8, and her value for any standalone item, either A or B, is only 1. The auctioneer initializes price clocks at $p(0) = (0, 0)$ and increments the clock price for any good with excess demand at a constant speed. For convenience, we assume that all bidders demand the smallest bundle when indifferent. All necessary details for this example are provided in Table 1.

Table 1: An Illustrative Example of the Ascending Auction

	Bidder 1	*Bidder 2*	*Bidder 3*		
Values (A, B, AB):	$v_1 = (3, 0, 3)$	$v_2 = (0, 10, 10)$	$v_3 = (1, 1, 8)$		
Efficient Allocation:	A	B	$\emptyset$		
Vickrey Payments:	1	5	0		
Clock Prices		*Bidding Information*		$Z(t)$	$Z(N, t)$
$t: 0 \to 3$	$x_1 = A$	$x_2 = B$	$x_3 = AB$	(1, 1)	(1, 1)
$p(t) = (t, t)$	$\widehat{\Delta}_1 = \{A, AB\}$	$\widehat{\Delta}_2 = \{B, AB\}$	$\widehat{\Delta}_3 = \{AB\}$		
	$\hat{v}_1 = (t, t, t)$	$\hat{v}_2 = (t, t, t)$	$\hat{v}_3 = (t, t, 2t)$		
$t: 3 \to 5$	$x_1 = \emptyset$	$x_2 = B$	$x_3 = AB$	(0, 1)	(0, 1)
$p(t) = (3, t)$	$\widehat{\Delta}_1 = \Omega$	$\widehat{\Delta}_2 = \{B, AB\}$	$\widehat{\Delta}_3 = \{AB\}$		
	$\hat{v}_1 = (3, 0, 3)$	$\hat{v}_2 = (3, t, t)$	$\hat{v}_3 = (3, t, 3 + t)$		
$t: 5 \to 7$	$x_1 = \emptyset$	$x_2 = B$	$x_3 = \emptyset$	(0, 1)	(0, 0)
$p(t) = (3, t)$	$\widehat{\Delta}_1 = \Omega$	$\widehat{\Delta}_2 = \{B, AB\}$	$\widehat{\Delta}_3 = \Omega$		
	$\hat{v}_1 = (3, 0, 3)$	$\hat{v}_2 = (3, t, t)$	$\hat{v}_3 = (1, 1, 8)$		
At termination:	$x_1 = \emptyset$	$x_2 = B$	$x_3 = \emptyset$	(0, 0)	(0, 0)
$T = 7$	$\widehat{\Delta}_1 = \Omega$	$\widehat{\Delta}_2 = \{B, AB\}$	$\widehat{\Delta}_3 = \Omega$		
$p(T) = (3, 7)$	$\hat{v}_1 = (3, 0, 3)$	$\hat{v}_2 = (3, 7, 7)$	$\hat{v}_3 = (1, 1, 8)$		

For this example, the development of the ascending auction can be decomposed into three stages. In stage one, clock prices rise from $p(0) = (0, 0)$ to $p(3) = (3, 3)$. At $p(3)$, Bidder 1 stops demanding item A and drops to $\emptyset$ which allows the auctioneer to elicit Bidder 1's value for A and AB. However, the value for item B is still missing, and the auctioneer asks Bidder 1 to report its marginal value for item B relative to its current demand.

At $p(3)$, there is no excess demand for item A and only the clock price for item B is rising in stage two. At $p(5) = (3, 5)$, Bidder 3 reduces its demand to $\emptyset$. At this time, the auctioneer knows that the value of Bidder 3 for AB is 8, but does not know its values for items A and B. The auctioneer asks Bidder 3 to report its marginal value for both item A and B relative to its current demand, and the value function of Bidder 3 is fully revealed. There are three observations that should be made at this point. First, the standard Walrasian aggregate demand at $p(5)$ is $(0, 1)$, already below the supply. The excess demand for the main economy is $Z(N, 5) = (0, 0)$ indicating that the main economy is cleared. However, the cumulative excess demand that accounts for the main economy and all marginal economies is still positive since the marginal economy for Bidder 1 is not yet cleared and the auction has to proceed further.

In the last stage, clock prices rise from $p(5) = (3, 5)$ to $p(7) = (3, 7)$. At $p(7)$, the marginal economy $E(N_{-1})$ is finally cleared, and the cumulative excess demand is zero. Bidder 1 is awarded item A and charged $y_1^V = \hat{v}_3(AB) - \hat{v}_2(B) = 1$. Bidder 2 gets item B and pays $y_2^V = \hat{v}_3(AB) - \hat{v}_1(A) = 5$. Note that while value functions of both Bidder 1 and Bidder 3 are fully elicited, Bidder 2 revealed its value function only partially.

4 Implementation Issues

In this section, we discuss several issues related to implementing the ascending auction described in Section 3.

4.1 Providing Feedback

One of the most important advantages associated with dynamic auctions is their ability to progressively inform bidders about the current level of competition, and their prospective winnings and payments. This is especially important for high-stake auctions where bidders need to have a sense of the current status to adjust their strategy for future rounds. For the ascending auction presented in Section 3, this can be relevant since the clock prices generally provide an imprecise estimate of the future payment.

An auctioneer can assist bidders by providing them with more relevant estimates. In terms of the auction outcome, a natural feedback would include informing bidder i about its current tentative assignment $x_i^*(N, t)$ and its current tentative payment calculated using (3.14). However, such feedback can be frustrating for bidder i if $x_i^*(N, t) \notin \widehat{\Delta}_i(t)$. A better approach would be to provide bidder i with its winning allocation in a constrained winner determination problem in which bidder i is restricted to win a bundle from $\widehat{\Delta}_i(t)$. The corresponding tentative Vickrey payment is obtained using (3.14) by replacing the solution for the main economy with the solution from the constrained problem.[21]

Most dynamic auctions inform bidders about the current level of competition in the auction via a disclosure of excess demand. This is natural for auctions where excess demand is used for incrementing clock prices, and therefore, is already partially revealed to bidders. For our auction, excess demand $Z(t)$ defined in (3.12) is the most fitting for these purposes as it matches the closing condition and the rule for incrementing clock prices. It might be tempting for the auctioneer to reveal excess demand for the main economy $z(N, t)$ since marginal economies are not relevant for determining the winning allocation. However, this is conceptually incorrect since the ultimate goal of the auction is implementing the Vickrey outcome in which allocations in marginal economies are equally important as the allocation in the main economy.

Providing feedback to bidders can also cause problems. Detailed feedback, such as reporting $z(N, t)$ or providing bidders with their tentative winnings and payments, can sometimes lead to incentive problems in environments where bidders can infer the current status of the main economy (i.e., whether it has been cleared). We provide a detailed discussion of this issue in the next section.

4.2 Indifferent Bidders

To implement the Vickrey outcome, the closing condition must ensure that both the main economy and all marginal economies are cleared at the time of termination. Due to stronger competition, it is likely for the main economy to be the last one to clear, but it is also possible that it is cleared ahead of some marginal economies (see example in Section 3.4). Vickrey payments do not depend on bidders' own bids, so once the main economy is cleared and the winning allocation is determined, all bidders become indifferent among all bids that are feasible to them.

When bidders are informed about the clearing of the main economy, their incentives for truthful bidding are compromised since their own bids do not affect their own payment but they do affect payments of other bidders. An indifferent bidder might decide to completely cease its bidding (a problem known as "quiet bidding") or, alternatively, bid in a way that inflates payments of other bidders (a problem known as "predatory bidding" or "spiteful bidding").

[21] A constrained winner determination problem is (2.3) with an additional constraint requiring $x_i \in \widehat{\Delta}_i(t)$. The corresponding payment for bidder i is calculated as $\sum_{j \in N_{-i}} \left[\hat{v}_j(x^*_j(N_{-i}, t)) - \hat{v}_j(\tilde{x}_j(N, t))\right]$ where $\tilde{x}(N, t) = (\tilde{x}_1(N, t), ..., \tilde{x}_n(N, t))$ solves the constrained optimization problem.

It is important to emphasize that the problem of indifferent bidders arises when bidders can learn about the status of the main economy via some channel. A good example of an auction design that can prematurely disclose the status of the main economy is the Combinatorial Clock Auction (CCA). For CCA, Levin and Skrzypacz (2016) show that the problem of indifference can lead to existence of inefficient equilibria. The same critique would apply to our ascending auction if bidders are informed about the status of the main economy. However, if the auctioneer uses a "cumulative version" of excess demand (like the one defined by (3.12)) for determining clock price increments, the status of the main economy would stay hidden until the end of the auction (see example in Section 3.4). One possible improvement to the proposed ascending auction would be an alternative way of calculating excess demand that targets marginal economies first and switches to targeting the main economy later.

Mishra and Parkes (2007) showed that restricting the preferences domain solves the problem with indifferent bidders for their auction design. In particular, the *buyers are substitutes* (BAS) condition guarantees that marginal economies always clear before the main economy in their auction; and the BAS condition is satisfied when the preferences domain is limited to gross substitutes.

Unfortunately, this result does not hold for our ascending auction. The reason for this is the difference between elicitation approaches. Mishra and Parkes (2007) use a semi-truthful elicitation approach where $\hat{v}_i(z,t) = v_i(z) - \alpha_i(t)$ for all $z \in \widehat{\Delta}_i(t)$ and $\hat{v}_i(z,t) = 0$ for all $z \notin \widehat{\Delta}_i(t)$. In contrast, the elicitation process in (3.8) must use a more aggressive estimate $p(t)\, z - gv_i(t,z) \geq 0$ for bundles $z \notin \widehat{\Delta}_i(t)$ to compensate for using linear prices. As a result, the BAS condition no longer guarantees the natural clearing order.

To illustrate this issue, consider an example with three items (one unit of good A and two units of good B) and three bidders. Bidder 1 values good A at 2 and has no value for good B, while Bidder 2 has no value for good A and values each unit of good B at 5. Bidder 3 has values $v_3(A) = 7$, $v_3(B) = 6$, $v_3(AB) = 10$ and $v_3(BB) = 9$. Note that these value functions satisfy the gross substitutes condition.

The dynamics of the ascending auction for this example is provided in Table 2. The main economy clears at $p(3) = (3,3)$ when approximations of the value functions are $\hat{v}_1 = (2,0,2,0)$, $\hat{v}_2 = (3,3,6,6)$ and $\hat{v}_3 = (3,3,6,6)$. Note that both $\hat{v}_2$ and $\hat{v}_3$ are not semi-truthful, and the payment calculation for Bidder 2 based on these approximations is 5 while her true Vickrey payment is only 4.

Intuitively, the ascending auction in this example must uncover the efficient allocation in the marginal economy $E(N_{-2})$ which includes Bidder 3 winning both units of good B. To elicit Bidder 3's value for this bundle, the auction should increment the clock price for good A at a faster rate than the one for good B.

Table 2: Example with Substitutes

	Bidder 1	*Bidder 2*	*Bidder 3*
Values for (A, B, AB, BB):	$(2, 0, 2, 0)$	$(0, 5, 5, 10)$	$(7, 6, 10, 9)$
Efficient Allocation:	$\emptyset$	BB	A
Vickrey Payments:	0	4	2

Clock Prices	$x_1(t)$	$x_2(t)$	$x_3(t)$	$z(N_{-1}, t)$	$z(N_{-2}, t)$	$z(N_{-3}, t)$	$z(N, t)$	$Z(t)$
$t : 0 \to 2$ $p(t) = (t, t)$	A	BB	AB	(0, 1)	(1, 0)	(0, 0)	(1, 1)	(1, 1)
$t : 2 \to 3$ $p(t) = (t, t)$	$\emptyset$	BB	AB	(0, 1)	(1, 0)	(0, 0)	(0, 1)	(1, 1)
$p(3) = (3, 3)$	$\emptyset$	BB	A	(0, 0)	(1, 0)	(0, 0)	(0, 0)	(1, 0)

But the auctioneer has no apparent way of knowing this since the main economy and the marginal economy $E(N_{-1})$ have excess demand for good B while only economy $E(N_{-2})$ has excess demand for good A. This example shows that the use of linear prices as the base for an elicitation process has a potential cost of clearing the main economy before marginal economies even in well-behaved settings.

4.3 Elicitation Burden

Another important implementation concern for dynamic auctions is the number of queries needed to elicit sufficient amount of information to implement the desired outcome. For general settings, ascending auctions proposed by Mishra and Parkes (2007) provide a comparative benchmark for testing the auction design proposed in Section 3. The comparative analysis can be carried out using simulations which would need to consider multiple variations of auction designs and an array of different value environments.

But even without simulations, we can identify settings in which our auction design performs well. For example, in a homogeneous environment and bidders with decreasing marginal values, our ascending auction simplifies to the ascending auction with "clinching" developed by Ausubel (2004) and delivers all advantages associated with using linear prices for elicitation. For another example, consider an environment with two goods, A and B, available in multiple quantities and bidders who view these goods as close substitutes but also put small premium on symmetric bundles (i.e., bundles where the total quantity is more equally distributed between the two goods).[22] In our design, the elicitation process reveals mostly symmetric bundles and avoids asymmetric bundles, such as the ones containing only A goods or only B goods. In contrast, the elicitation process in Mishra and Parkes (2007) uncovers the relative values of bundles strictly in the descending order of their value,[23] and the revelation of valuable but asymmetric bundles cannot be avoided.

In a general setting, the elicitation process from Section 3 can require bidders to report their marginal values for a large number of bundles at a time. In practice, the auctioneer should invite bidders to submit marginal values only for bundles of their interest and use zero incremental values for bundles that do not receive bids. In addition, the auctioneer should avoid unnecessary elicitation by eliminating bundles that can be proven irrelevant by the time their value has to be reported. A bundle is irrelevant for a given bidder if it can be shown that assigning it to the bidder leads to inefficient allocations in all relevant economies. The inefficiency of allocation x for economy $E(M)$ can be proved as follows:

Proposition 6. *A feasible allocation $x = (x_1, ..., x_n)$ is inefficient for economy $E(M)$ if there exists a feasible allocation $y = (y_1, ..., y_n)$ and time t such that*

$$y_i \in \Delta_i(t) \quad \forall\, i \in M \qquad \text{and} \qquad \textstyle\sum_M \hat{v}_j(x_j, t) \;<\; \sum_M \hat{v}_j(y_j, t). \tag{4.1}$$

Proposition 6 provides a direct way to test bundles for irrelevance during the auction by replacing $\Delta_i(t)$ with $\widehat{\Delta}_i(t)$. The full description of a test for irrelevance is provided in the appendix.

4.4 Discrete Prices

In the ascending auction described in Section 3, we used an intra-round bidding approach to accommodate a continuous price path. However, most dynamic auctions used in applications utilize discrete prices, generally with large increments.[24] Here we provide a description of our ascending auction that works with any discrete non-decreasing price path.

[22] This setting appears to fit the Ireland Multi-Band Spectrum Auction held in 2012 where bidders valued symmetric holdings in 800 MHz and 900 MHz bands.

[23] In order to infer the relative value of its 10^{th} most preferred bundle, the bidder would have to reveal relative values for its 1^{st}-9^{th} most preferred bundles.

[24] Dynamic auctions that use discrete prices (for example, Ausubel, 2004 and 2006; and Mishra and Parkes, 2007) restrict the value function domain to integers and use small discrete increments to make sure that the procedure properly captures all necessary value information. In other words, they use a discrete price path that works as "continuous" on the restricted value domain. The main disadvantage of such approach is the requirement for sufficiently small increments.

In the ascending auction of Section 3, the continuity of the price path was used to ensure that bidder i, when switching its demand at time t from $x_i(t-0)$ to $x_i(t)$, is indifferent between the old and new demands given prices $p(t)$, i.e.,

$$v_i(x_i(t-0)) - p(t)\, x_i(t-0) \;=\; v_i(x_i(t)) - p(t)\, x_i(t). \tag{4.2}$$

With discrete prices, this property is lost, but the auctioneer can collect an extra piece of information from a switching bidder to restore its equivalent.

Suppose that the auctioneer announces a price vector $p(t)$ and bidder i replies with its demand $x_i(t)$ at each time $t \in \{0, 1, 2, ...\}$. In addition, at each time $t \geq 1$, the auctioneer asks bidder i to report the smallest discount on the price of bundle $x_i(t-1)$, denoted $d_i(t)$, that bidder i commands to demand bundle $x_i(t-1)$ at $p(t)$ instead of its current demand $x_i(t)$.[25] When bidder i bids truthfully, discount $d_i(t)$ is given by:

$$d_i(t) \;=\; \big[v_i(x_i(t)) - p(t)\, x_i(t)\big] \;-\; \big[v_i(x_i(t-1)) - p(t)\, x_i(t-1)\big]. \tag{4.3}$$

This new piece of information is sufficient to construct the value approximation function $\hat{v}_i(., t)$ under discrete prices. The procedure for constructing the set of revealed bundles $\widehat{\Delta}_i(t)$ does not change, and all required modifications to activity rules and other key formulas can be found in the appendix.

5 Conclusion

The virtues of the VCG mechanism create a strong interest in dynamic auctions that can implement the VCG outcome. The older auction literature provided a number of practical designs for simple settings. For the general private value setting, Mishra and Parkes (2007) developed a dynamic auction that achieves the VCG outcome by relying on demand queries and non-linear and non-anonymous prices — a limitation that deems the design impractical.

In this paper, we show that flexible elicitation tools can avoid complex prices and facilitate a simpler auction process. We design an efficient ascending auction that uses an elicitation process based on linear and anonymous prices. To the best of our knowledge, it is the first dynamic auction that uses simple and practical elicitation methods to implement the Vickrey outcome in the general private value setting. The key idea is using a combination of demand queries and marginal value queries to ensure that all potentially important information is captured properly. Several implementation issues, including a version of the design based on discrete prices, are discussed.

[25] Note that in practice, the auctioneer needs to ask for $d_i(t)$ only when bidder i demands bundle $z \notin \widehat{\Delta}_i(t-1)$. For $z \in \widehat{\Delta}_i(t-1)$, activity rule AR1 would limit $d_i(t)$ to a single number.

A Appendix

A.1 Proofs

PROOF OF THEOREM 1. Suppose that all bidders bid truthfully. Given the adjustment rule and an iterative elicitation process, the procedure would eventually force each bidder to fully reveal its $v(.)$ at which point $\widehat{\Delta}_i(t) = \Omega$ for all $i \in N$ and the closing condition is satisfied. Therefore, the procedure has to end in finite time T. For a straightforward elicitation process, $x^*(N, T)$ is efficient by Proposition 1 and $p_i^V = y_i^V$ by Proposition 2. Now suppose that bidder i deviates from truthful bidding. First, note that bidder i cannot block the auction from closing. Second, denote $\tilde{v}_i(z) = \hat{v}_i(z, T)$ which is well-defined. Due to a straightforward elicitation process, $\widehat{\Delta}_j(T) \subseteq \Delta_j(T)$ for all bidders in N_{-i}, and $\widehat{\Delta}_i(T) \subseteq \tilde{\Delta}_i(T)$ where $\tilde{\Delta}_i(T)$ is the analog of $\Delta_i(T)$ for $\tilde{v}_i(.)$. By Propositions 1 and 2, the auction outcome corresponds to a Vickrey outcome for the value profile $(\tilde{v}_i, v_{-i})$ which is weakly dominated for bidder i. □

PROOF OF PROPOSITION 3. (Necessity) For AR1, suppose that bidder i bids truthfully according to $\tilde{v}_i(.)$ and z is her true demand at time t. Then $gv_i(t, z) = 0$ by the Afriat's theorem. If $z \geq x_i(s)$ for some time $s < t$, then it must be the case that z and $x_i(s)$ are both optimal demands at time t. For AR2, by revealed preference,

$$\begin{aligned} \tilde{v}_i(z) &\leq \min_{s\in[0,t]} \{\tilde{v}_i(x_i(s)) + p(s)[z - x_i(s)]\} \\ &= \min_{s\in[0,t]} \left\{\tilde{v}_i(x_i(t)) - \textstyle\int_s^t p(u)dx_i(u) + p(s)[z - x_i(s)]\right\} \\ &= \tilde{v}_i(x_i(t)) + p(t)[z - x_i(t)] - gv_i(t, z) \end{aligned}$$

and the marginal value between bundles z and $x_i(t)$ is given by $\tilde{v}_i(z) - \tilde{v}_i(x_i(t)) \leq p(t)[z - x_i(t)] - gv_i(t, z) = \widetilde{mv}_i(z)$. Then AR2 never overconstrains a bidder who bids according to some value function $\tilde{v}_i(.)$. *(Sufficiency)* Suppose that both AR1 and AR2 hold on $[0, t]$. Construct $\tilde{v}_i(.)$ as

$$\tilde{v}_i(z) = \begin{cases} p(t)\, x_i(t) - \int_{t(z)}^t p(u)dx_i(u) + \widetilde{mv}_i(z) & z \in \widehat{\Delta}_i(t) \\ 0 & z \notin \widehat{\Delta}_i(t) \end{cases}$$

where $t(z) = s'$ and $\widetilde{mv}_i(z) = 0$ if bundle z was demanded by bidder i at time $s' \leq t$. Now we show that $\tilde{v}_i(.)$ rationalizes bidding of bidder i at any time $s \in [0, t]$. For bundle z that was demanded at some time $s' \in [0, t]$ (i.e., $x_i(s') = z$), the bidding on $[0, t]$ is rationalized since $\tilde{v}_i(z) - p(s)\, z \leq \tilde{v}_i(x_i(s)) - p(s)\, x_i(s)$ is equivalent to $p(s)[x_i(s) - z] + \int_s^{s'} p(u)dx_i(u) \leq 0$ which is satisfied due to AR1. For bundle $z \in \widehat{\Delta}_i(t)$ that was never demanded explicitly but was added to the set of revealed bundles at time $t(z)$ (so $z \geq x_i(t(z))$):

- For $s \leq t(z)$, the bidding on $[0,t]$ is rationalized since $\tilde{v}_i(z) - p(s)\,z \leq \tilde{v}_i(x_i(s)) - p(s)\,x_i(s)$ is equivalent to $\widetilde{mv}_i(z) \leq -p(s)[x_i(s) - z] - \int_s^{t(z)} p(u)dx_i(u) \leq p(t(z))[z - x_i(t(z))] - gv_i(t(z), z)$ which is satisfied due to AR2.

- For $s > t(z)$, we show that $gv_i(t(z), z) = gv_i(s, z)$ (i.e., the GARP violation for bundle z is not going to increase over time). Then the bidding is rationalized by the same argument as for the case $s \leq t(z)$. Suppose that there exist $s' \in (t(z), s]$ such that the GARP violation for bundle z is strictly higher for s' than for $t(z)$ (i.e., $p(s')[x_i(s') - z] + \int_{s'}^{s} p(u)dx_i(u) > p(t(z))[x_i(t(z)) - z] + \int_{t(z)}^{s} p(u)dx_i(u)$) But then $p(s')[x_i(s') - x_i(t(z))] + \int_{s'}^{t(z)} p(u)dx_i(u) > [p(s') - p(t(z))][z - x_i(t(z))] \geq 0$ where the last inequality implies violation of AR1 at time s'. Hence, $gv_i(t(z), z) = gv_i(s, z)$.

□

Lemma 1. *If bidder i bids truthfully according to $v_i(.)$ on $[0,t]$, then for all $z \in \widehat{\Delta}_i(t)$, the revealed marginal value of bundle z relative to bundle $x_i(t)$ is the true marginal value, i.e., $mv_i(z,t) = v_i(z) - v_i(x_i(t))$.*

Proof. First, suppose that there exists $t' \in [0,t)$ such that $x_i(t') = z$. Denote $t_1, ..., t_m$ all times in the interval $[t', t]$ when bidder i changed its demand. Given the continuous price path $p(.)$, bidder i who bids truthfully is indifferent at all switch points, i.e.,

$$\begin{aligned} v_i(z) - p(t_1)\,z &= v_i(x_i(t_1)) - p(t_1)\,x_i(t_1) \\ \dots \quad &\dots \quad \dots \\ v_i(x_i(t_{m-1})) - p(t_m)\,x_i(t_{m-1}) &= v_i(x_i(t)) - p(t_m)\,x_i(t) \end{aligned}$$

Then by (3.7) $v_i(z) = v_i(x_i(t)) - \int_{t'}^{t} p(u)dx_i(u) = v_i(x_i(t)) + mv_i(z,t)$. Second, if $z \in \widehat{\Delta}_i(t)$, but it was never explicitly demanded by bidder i, then by (3.7) and the definition of truthful bidding, we have $mv_i(z,t) = \int_t^{t(z)} p(u)dx_i(u) + \widetilde{mv}_i(z) = v_i(z) - v_i(x_i(t))$. □

PROOF OF PROPOSITION 4. Formulas in (3.1) and (3.8) indeed represent an elicitation process since they uniquely define $\widehat{\Delta}_i(t)$ and $\hat{v}_i(., t)$ at each time t and each bidder i. By construction, the elicitation process is iterative and ascending since $p(.)$ is nondecreasing. (Straightforward) Bidder i bids truthfully according to $v(.)$ on $[0,t]$. For any bundle $z \in \widehat{\Delta}_i(t)$, $\delta_i(z,t) = \delta_i(x_i(t), t)$ by Lemma 1. For any bundle $z \notin \widehat{\Delta}_i(t)$, $\delta_i(z,t) \leq \delta_i(x_i(t), t)$ by (3.5). Then if $z \in \widehat{\Delta}_i(t)$, then $z \in \Delta_i(t)$. (Monotonic) Consider time $t' < t$ and bundle $z \notin \widehat{\Delta}_i(t)$. Then $\hat{v}_i(z,t) - \hat{v}_i(z,t') = [p(t) - p(t')]\,z - [gv_i(t,z) - gv_i(t',z)] \leq \hat{v}_i(x_i(t'), t) - \hat{v}_i(x_i(t'), t')$. But then $\hat{v}_i(z,t) - \hat{v}_i(z,t') \leq \hat{v}_i(y,t) - \hat{v}_i(y,t')$ for any $y \in \widehat{\Delta}_i(t')$. □

PROOF OF PROPOSITION 5. If economy $E(M)$ is cleared at time t, then $0 \in Z(M,t)$ by the definition of $Z(M,t)$. For the converse, suppose that $0 \in Z(M,t)$, but $E(M)$ is not cleared at t. Then for any tentative assignment $x^*(M,t)$, there is at least one bidder $i \in M$ such that $x_i^* \notin \widehat{\Delta}_i(t)$. For such bidder i, there is at least one good for which demand $d \in D_i(t)$ strictly exceeds the tentative award x_i^* since otherwise, $x_i^* \geq d$ and $x_i^* \in \widehat{\Delta}_i(t)$. Then $0 \notin Z(M,t)$. □

PROOF OF THEOREM 2. By Proposition 4, the elicitation process is straightforward and iterative. For each economy $E(M)$ that is not cleared at integer time t, $Z^k(t) > 0$ if product k is overdemanded in at least one relevant economy, and its price will be increased by $\epsilon\, p^k(t) > 0$ on the time interval $(t, t+1]$. Increasing clock prices for overdemanded goods cause an increase in $\hat{v}_i(z,t)$ for all bundles $z \in \widehat{\Delta}_i(t)$ for bidder i who demands overdemanded item at time t (satisfies the adjustment rule (2.9) on a sufficiently long time interval). Then the Vickrey outcome is implemented by Theorem 1. □

PROOF OF PROPOSITION 6. A feasible allocation x is inefficient since:

$$\sum_{j\in M} v_j(x_j) \;=\; \sum_{j\in M}[\hat{v}_j(x_j,t)+\delta_j(x_j,t)] \;<\; \sum_{j\in M}[\hat{v}_j(y_j,t)+\delta_j(y_j,t)] \;=\; \sum_{j\in M} v_j(y_j)$$

□

A.2 Test for Irrelevant Bundles

For economy $E(M)$ and bidder $i \in M$, the following steps have to be carried out to test bundle z for irrelevance at time t:

1. Find the maximal value that can be obtained by giving each bidder in M a bundle from its set of revealed bundles $\widehat{\Delta}_i(t)$ by solving the following problem:

$$H1 \;=\; \max_{y\,\in X} \textstyle\sum_{j\in M} \hat{v}_j(y_j,t) \qquad \text{s.t.} \quad y_j \in \widehat{\Delta}_j(t) \qquad \forall j \in M \qquad \text{(A.1)}$$

   If solution does not exist, the test cannot proceed to step 2.

2. Find the maximal value that can be obtained by allocating bundle z to bidder i by solving the following problem:

$$H2 \;=\; \max_{x\in X} \textstyle\sum_{j\in M/\{i\}} \hat{v}_j(x_j,t) \qquad \text{s.t.} \qquad x_i = z \qquad \text{(A.2)}$$

   If $p(t)\, z \,-\, gv_i(z,t) \;\leq\; H1 \,-\, H2$, then bundle z is irrelevant for bidder i in economy $E(M)$.

A.3 Discrete Version of the Ascending Auction

In this section, we provide required modifications to implement the discrete version of the ascending auction described in Section 4.4. The formulas that are not mentioned here carry over from Section 3.

First, we modify the definition of truthful bidding. We say that bidder i *bids truthfully according to its value function* $v_i(.)$ at time $t \in \{0, 1, 2, \dots\}$ if:

(a) Bidder i truthfully reports its demand-discount pair $(x_i(t), d_i(t))$ given $p(t)$:

$$\begin{aligned} x_i(t) &\in \arg\max_{z \in \Omega} \big[v_i(z) \; - \; p(t)\, z\big]; \\ d_i(t) &= \big[v_i(x_i(t)) - p(t)\, x_i(t)\big] \; - \; \big[v_i(x_i(t-1)) - p(t)\, x_i(t-1)\big]. \end{aligned} \tag{A.3}$$

(b) (same as for the continuous version)

For bidder i, denote $dx_i(t) = x_i(t) - x_i(t-1)$ a change in demand at time t, and $dv_i(t) = p(t)\, dx_i(t) + d_i(t)$ an implied change in value at time t. Note that $dv_i(t) = v_i(x_i(t)) - v_i(x_i(t-1))$ when bidder i bids truthfully. Then the discrete analog of the GARP condition is given by:

$$(GARP) \qquad p(s)[x_i(s) \; - \; x_i(s')] \; + \; \sum_{u=s'}^{s+1} dv_i(u) \; \leq \; 0 \qquad \forall \; s < s' \in \{0, ..., t\}.$$

and the discrete analog of the GARP violation is given by:

$$gv_i(t, z) \; = \; \max_{s \in \{0,\dots,t\}} \left\{ p(s)[x_i(s) - z] \; + \; \sum_{u=t}^{s+1} dv_i(u) \; + \; p(t)[z - x_i(t)] \right\}. \tag{A.4}$$

The modified activity rule AR1: At any time t, a bundle-discount pair $(z, d) \in \Omega \times \mathbb{R}$ is unacceptable if either $gv_i(t, z) > 0$ or $z \geq x_i(s)$ for some time $s < t$ (when validating AR1, $gv_i(t, z)$ is calculated using (A.4) by setting $x_i(t) := z$ and $d_i(t) := d$). Activity rule AR2 does not change. The discrete analog of formula (3.7) is given by:

$$mv_i(z, t) \; = \; \begin{cases} -\sum\limits_{u=t}^{t'+1} dv_i(u) & \text{if } \; \exists\, t' \in \{0, ..., t\} : \; x_i(t') = z \\ -\sum\limits_{u=t}^{t(z)+1} dv_i(u) \; + \; \widetilde{mv}_i(z) & \text{otherwise} \end{cases}. \tag{A.5}$$

Finally, the price adjustment formula is modified as follows:

$$p^k(t) \; = \; \begin{cases} p^k(t-1)\,[1 \; + \; \epsilon] & \text{if} \quad Z^k(t-1) > 0 \\ p^k(t-1) & \text{if} \quad Z^k(t-1) = 0 \end{cases}. \tag{A.6}$$

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