Efficient Dynamic Auctions for Private Valuations*

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Abstract

The literature on dynamic implementation of the VCG mechanism routinely uses price - demand queries for preference elicitation. As a result, all known dynamic auction procedures that implement the Vickrey outcome for the general private valuation domain are impractical. The paper makes two contributions. First, we show that the general problem of designing a dynamic VCG mechanism can be reduced to a problem of specifying an appropriate elicitation process that satisfies certain properties. Second, we demonstrate our main result by characterizing an ascending auction that implements the Vickrey outcome for general private valuation domain while using a single linear and anonymous price trajectory for preference elicitation. To the best of our knowledge, it is a first dynamic auction design that uses a familiar and practical elicitation process to produce the Vickrey outcome.

Keywords: Combinatorial auctions, Multi-item auctions, Vickrey auctions, Dynamic Auctions, Ascending Auctions, Spectrum Auctions

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Auction design aims to develop market mechanisms for allocating scarce and valuable resources. While there are many objectives that the auctioneer might pursue, in public auctions the auctioneer routinely targets *efficiency of the allocation* (i.e., awarding items to parties who value them the most) as its primary goal. Since the famous Vickrey-Clarke-Groves (VCG) mechanism[1] is essentially the unique mechanism that guarantees efficiency of the allocation in a dominant strategy sense without requiring any transfers by losing bidders, the VCG mechanism has been the focal point of the auction design literature for decades.[2]

Dynamic implementations of the VCG mechanism are of special interest due to the general perception that dynamic auctions are preferred over their sealed-bid counterparts in many applications. The standard list of advantages includes privacy preservation, learning of interdependencies, budget constraints, corporate governance and etc. The literature on designing efficient dynamic auctions via implementing the VCG outcome is abundant (see for example, Demange et al. (1986), Gul and Stacchetti (2000), Parkes and Ungar (2000 and 2002), Ausubel and Milgrom (2002), Bikhchandani and Ostroy (2002 and 2006), Ausubel (2004 and 2006), de Vries et al. (2007), Mishra and Parkes (2007), Lamy (2012) and Baranov et al. (2016)). The excellent survey of the literature is provided by Parkes (2006).

For the general setting with private values, Mishra and Parkes (2007) describe an ascending auction that implements the VCG outcome via a single price path. The design is centered around the concept of *universal competitive equilibrium* (UCE) prices. In fact, the UCE prices are shown to be necessary and sufficient for dynamically implementing the VCG outcome. Unfortunately, the UCE prices have to be rather complex including nonlinearities and nonanonymity (bundle-specific and bidder-specific prices). Such complexity significantly limits the practical appeal of their design. In this paper, we design an efficient ascending auction that avoids some of these complexities.

Our main contribution is a general decomposition result. Using a general framework, we show that the problem of designing a dynamic VCG mechanism can be decomposed into two parts: (1) specifying an appropriate elicitation process for a given valuation domain; and (2) running ascending Vickrey auction (adjustment rule, closing rule, and outcome rule) based on the specified elicitation process.

The decomposition result has two immediate implications. First of all, it generalizes the standard results for implementing the VCG outcome in various settings as an ex post equilibrium.[3] Second, it allows the auctioneer to focus on

[1] See Vickrey (1961), Clarke (1971) and Groves (1973).

[2] See Green and Laffont (1979) and Holmstrom (1979) for the uniqueness result.

[3] See Gul and Stacchetti (2000), Ausubel (2004 and 2006), Bikhchandani and Ostroy (2006), de Vries et al. (2007) and Mishra and Parkes (2007).

designing an elicitation process suitable for its application instead of designing a compete auction. Since any appropriate elicitation process can be used as a base for constructing a dynamic VCG mechanism, the auctioneer can specify a custom elicitation process that takes advantages of the application-specific setting.

Our second result is a full specification of an efficient ascending auction for general valuations that implements the VCG outcome as an ex post equilibrium using a *single linear and anonymous price trajectory* as the base elicitation tool. To circumvent the negative results about insufficiency of linear and anonymous prices,[4] we expand the set of queries about values that can be made by the auctioneer in cases when the linear and anonymous prices miss important information. To put it differently, we depart from the standard "Walrasian auctioneer" elicitation protocol that restricts the auctioneer to elicit value information via price-demand queries.

To illustrate our simple approach, assume a homogeneous setting with three identical items and a bidder with the following values $v(1) = 20$, $v(2) = 25$, $v(3) = 40$. Suppose that the auctioneer uses a standard elicitation protocol by quoting a per unit price and asking bidders to report their demand. In this example, the auctioneer cannot elicit bidder's value for 2 units since its demand is 3 units for any $p \leq 10$ and 1 unit for any $p > 10$. If the value for two units happens to be important for achieving efficiency, the linear prices are insufficient. Now suppose that the auctioneer can ask extra questions about any missing information. For example, the auctioneer can ask bidders for marginal value information for any missed quantities at the time their demand drops by more than 1 unit. In our example, when the bidder lowers its demand from 3 units to 1 unit at $p = 10$, the auctioneer can simply ask the bidder to report its marginal value for 2 units (e.g., $v(2) - v(1) = 5$).

Historically, the literature on dynamic implementation of the VCG mechanism is centered around the definition of the ascending auction. In fact, Gul and Stacchetti (2000, pp. 1) provide their definition of a dynamic auction in the very first sentence of their paper:

> *"A dynamic auction can be described as a rule for adjusting prices given the observed history of demand (i.e., bids) and a rule for terminating the price adjustment procedure and specifying an allocation (i.e., determining who gets the good(s) and at what price(s)). The English auction is also identified with the property that prices are non-decreasing. More specifically, the English auction is typically identified with the procedure of increasing the prices as long as there is excess demand."*

[4]See Gul and Stacchetti (2000), Bikhchandani and Ostroy (2002), and Mishra and Parkes (2007).

This rather strict definition was subsequently relaxed: (1) to allow multiple price trajectories [Ausubel (2006)]; (2) to allow discounts on prices for final payment calculations [Ausubel (2006) and Parkes and Mishra (2007)]; (3) to allow nonlinear and nonanonymous price trajectory [de Vries et al. (2007) and Mishra and Parkes (2007)]; and (4) to allow a general closing rule that postpones the termination until the efficient allocation in all relevant economies has been established [Mishra and Parkes (2007)]. In this spirit, our paper puts forward another relaxation for the dynamic auction – the auctioneer is not bound to use price-demand elicitation process.

It is worth noting that many auction designs that are implemented in practice have already dropped the restrictive price-demand elicitation tool and embraced more flexible ways of preference elicitation. For example, Combinatorial Clock Auction (CCA), an auction design that has been recently used for many spectrum auctions worldwide. CCA uses the standard price-quantity queries during the initial phase of the auction. However, CCA supplements the information with an extra round at the end (referred to as the supplementary round) where the price-demand protocol is replaced with a direct elicitation of values for a large number of bundles. In 2016, a new auction format, Combinatorial Multi-Round Ascending Auction (CMRA) has been used to allocate 1800 MHz spectrum in Denmark. In CMRA, the auctioneer quotes linear and anonymous prices and bidders reply with their demand, but at the same time, bidders are also allowed to specify alternative bids. While not implemented in practice at the time of this writing, Baranov et al. (2016) describe a new elicitation procedure suitable for procurement auctions with increasing returns. Under their elicitation procedure, suppliers are asked to list all profitable quantities given the per unit price instead of very uninformative question about their supply.

The article is organized as follows. Section 1 provides a model of the environment. The general framework for dynamic Vickrey auctions and the decomposition result are presented in Section 2. The new efficient ascending auction is described in Section 3. Section 4 presents a detailed example of the new auction design. Several issues related to applications are discussed in Section 5. Section 6 concludes. Some of the proofs are relegated to Appendix A.

1 Model

A seller offers multiple units of K heterogeneous indivisible goods, denoted as $S = \{s^1, ..., s^K\} \in Z^K_{++}$ to a set of bidders $N = \{1, ..., n\}$. The set of all possible bundles of items in S is denoted by $\Omega = \{(z^1, ..., z^K) : 0 \leq z^k \leq s^k \quad \forall k \in \{1, ..., K\}\}$. For every bidder $i \in N$, and every bundle $z \in \Omega$, the valuation of bidder i is given by $v_i(z) \geq 0$. A bidder's value for the null bundle is normalized to zero, $v_i(0) = 0$. We make the following standard assumptions about valuation functions:

(A1) *Pure Private Values:* Each bidder i knows its own valuation for any bundle z, and this valuation does not depend on valuations of other bidders;

(A2) *Quasilinear Values:* The payoff of bidder i from winning bundle z in exchange for a payment y is given by $v_i(z) - y$;

Another typical restriction made in the literature is monotonicity (free disposal).[5] This assumption is not needed for our results.

An allocation $x = (x_1, ..., x_n)$ is called *feasible* if $x_i \in \Omega$ for all $i \in N$ and $\sum_N x_j \leq S$. Denote X a set of all feasible allocations. The *coalitional value function* for bidders in coalition $M \subseteq N$ is given by:

$$w(M) \;=\; \max_{x \in X} \sum_{j \in M} v_j(x_j) \tag{1.1}$$

We denote $E(M)$ an economy that only includes bidders in $M \subseteq N$. A feasible allocation $x = (x_1, ..., x_n) \in X$ is *efficient* for economy $E(M)$ if

$$\sum_M v_j(x_j) = w(M). \tag{1.2}$$

Let $N_{-i} = N \backslash \{i\}$ denote the coalition of all bidders in N excluding bidder i. A *Vickrey outcome* consists of an efficient allocation vector $x^* = (x_1^*, ..., x_n^*)$ for the main economy $E(N)$ and a corresponding payment vector $y^V = (y_1^V, ..., y_n^V)$ where $y_i^V = w(N_{-i}) - \sum_{N_{-i}} v_j(x_j^*)$ for all $i \in N$. A Vickrey payoff for bidder i is given by $\pi_i^V = v_i(x_i^*) - y_i^V = w(N) - w(N_{-i})$.

2 Dynamic Vickrey Auction

In this section we describe a general class of dynamic Vickrey auctions. We start with the general framework and restatement of some known results.

2.1 Preliminaries

A dynamic auction has to reconstruct sufficient amount of bidders' value functions to establish efficiency of the resulting allocation. To achieve efficiency, the

[5] *Monotonicity:* The value function $v_i(z)$ is weakly increasing in z, i.e., $v_i(z') \geq v_i(z)$ for any $z' \geq z$.

auction uses a process for iterative preference elicitation and reconstruction of value functions. The majority of elicitation processes are designed after the famous *"Walrasian auctioneer"* [Walras (1874)] where the auctioneer quotes prices and asks bidders to report their demand at these prices (e.g., demand queries). However, an elicitation protocol that limits the auctioneer to demand queries is restrictive. For our purposes, we allow more general elicitation protocols which can use demand queries as well as any other queries for elicitation.

Suppose that the auctioneer starts the auction at $t = 0$. At each time $t \geq 0$, the auctioneer asks bidders to provide some information about their preferences, and, utilizing bidder i's responses that were received on the time interval $[0, t]$, the auctioneer constructs an approximation of bidder i's value function denoted by $\hat{v}_i(., t)$. Here we assume that the elicitation process produces a unique $\hat{v}_i(z, t)$ for each bundle $z \in \Omega$ and time t (also normalize $\hat{v}_i(0, t) = 0$ for any t). For each bidder i, denote the current approximation error for bundle z at time t as

$$\delta_i(z, t) = v_i(z) - \hat{v}_i(z, t) \tag{2.1}$$

and let $\Delta_i(t)$ denote a set of bundles with the highest approximation error at time t, i.e.:

$$\Delta_i(t) = \arg\max_{z \in \Omega} \delta_i(z, t) \tag{2.2}$$

By construction, $\delta_i(z, t) \geq 0$ for any bundle $z \in \Delta_i(t)$ since the null bundle $z = 0$ belongs to Ω and $\delta_i(0, t) = 0$. Note that set $\Delta_i(t)$ coincides with bidder i's demand correspondence at time t when the price of each bundle z is given by $\hat{v}_i(z, t)$.

Using the current approximation of value function for each bidder, the auctioneer can find $x^*(M, t)$, a tentative value-maximizing allocation for economy $E(M)$ at time t, by solving the standard winner determination problem:[6]

$$x^*(M, t) \in \arg\max_{x \in X} \sum_{j \in M} \hat{v}_j(x_j, t) \tag{2.3}$$

Proposition 1 below is a restatement of the result obtained by Parkes (2002) and Nisan and Segal (2006) using our notation.[7] It provides a sufficient condition to establish efficiency of tentative allocation $x^*(M, t)$ for economy $E(M)$.

Proposition 1 (Parkes (2002), Nisan and Segal (2006))**.** *A tentative allocation $x^*(M, t) = (x_1, ..., x_n)$ is efficient for economy $E(M)$ if*

$$x_i \in \Delta_i(t) \qquad \forall i \in M \tag{2.4}$$

[6]If the value-maximizing allocation is not unique, the auctioneer can pick one of them using some extra criteria.

[7]Parkes (2002) and Nisan and Segal (2006) provide the necessary and sufficient conditions. For our framework, only the sufficiency part can be established.

Proof. Observe that

$$\max_{x \in X} \sum_{j \in M} v_j(x_j) \quad \Leftrightarrow \quad \max_{x \in X} \sum_{j \in M} \left[\hat{v}_j(x_j, t) + \delta_j(x_j, t)\right].$$

But then, if allocation x solves (2.3) and condition (2.4) is satisfied, x must be efficient for $E(M)$. □

Proposition 1 provides the auctioneer with a simple and natural way to design an elicitation process that can identify an efficient allocation with partial knowledge of value functions. If the auctioneer constructs approximation of value functions in a way that informs him about the bundles in $\Delta_i(t)$ for each bidder i, the auctioneer can confirm the efficiency of the tentative allocation by verifying condition (2.4).

The next proposition establish the necessary and sufficient condition for an elicitation process to identify the Vickrey outcome. The proposition is a restatement of the Theorem 1 from Mishra and Parkes (2007). We provide a short proof for this result using our notation.

Proposition 2 (Mishra and Parkes (2007))**.** *Suppose that at time t, condition (2.4) is satisfied for the main economy $E(N)$. Then a Vickrey outcome can be identified from $\{\hat{v}_j(.,t)\}_{j\in N}$ if and only if at time t condition (2.4) is also satisfied for all marginal economies $\{E(N_{-1}), E(N_{-2}), ..., E(N_{-n}\})$.*

Proof. (Sufficiency) Suppose that condition (2.4) is satisfied for both the main economy $E(N)$ and the marginal economy for bidder i, $E(N_{-i})$, at time t. Then allocation $x = x^*(N,t)$ is efficient for $E(N)$ and allocation $x' = x^*(N_{-i},t)$ is efficient for $E(N_{-i})$ by Proposition 1 and

$$\begin{aligned} y_i^V &= \sum_{j\in N_{-i}} \left[v_j(x'_j) - v_j(x_j)\right] \\ &= \sum_{j\in N_{-i}} \left[\hat{v}_j(x'_j,t) - \hat{v}_j(x_j,t)\right] + \sum_{j\in N_{-i}} \left[\delta_j(x'_j,t) - \delta_j(x_j,t)\right] \\ &= \sum_{j\in N_{-i}} \left[\hat{v}_j(x'_j,t) - \hat{v}_j(x_j,t)\right] \end{aligned} \tag{2.5}$$

where the last equality follows from the fact that both x'_j and x_j belong to $\Delta_j(t)$ for each bidder $j \in N_{-i}$.

(Necessity) Suppose that $x'_j = x^*_j(N_{-i},t) \notin \Delta_j(t)$ for some bidder j. But then by (2.5),

$$y_i^V < \sum_{j\in N_{-i}} \left[\hat{v}_j(x'_j,t) - \hat{v}_j(x_j,t)\right].$$

□

Propositions 1 and 2 highlight the important role of condition (2.4) for dynamic implementations of the VCG mechanism. The only way to find the Vickrey outcome using approximations of value functions $\{\hat{v}_j(., t)\}_{j\in N}$ instead of true value functions $\{v_j(.)\}_{j\in N}$ is to uncover enough information such that condition (2.4) holds for all relevant economies. However, direct verification of condition (2.4) is infeasible for the auctioneer since the set $\Delta_i(t)$ is based on $v_i(.)$. Therefore, the elicitation process must construct both the approximation of the value function $\hat{v}_i(., t)$ and a feasible approximation for the set $\Delta_i(t)$.

2.2 Elicitation Process

In this section we put additional structure on the elicitation procedures that can be used as a base for a dynamic implementation of the VCG mechanism. We start with a mild technical restriction. An elicitation process is called *fully expressive* if the types of queries that the auctioneer can make are sufficient to enable elicitation of any value function that satisfies (A1) - (A2) from a bidder who truthfully responds to all queries.[8] The term "fully expressive" is standard in the literature on bidding languages (see Nisan (2006)). We limit our attention to fully expressive elicitation processes.

Now we provide a formal definition of an elicitation process. The definition emphasizes the role of two key elements of an elicitation process: current approximation of the value function, $\hat{v}_i(., t)$ and current approximation of the corresponding set $\Delta_i(t)$, denoted as $\widehat{\Delta}_i(t)$.

Definition 1. *An* ***elicitation process*** *is a procedure that for each time $t \geq 0$ specifies informational queries addressed to each bidder and converts bidder i's responses received on $[0, t]$ into a single-valued function $\hat{v}_i(., t)$ and a set of bundles $\widehat{\Delta}_i(t)$.*

This definition is less innocent than it sounds. It assumes that the elicitation procedure is sophisticated enough to resolve any inconsistencies in bidders' responses to always produce a well-defined and single-valued $\hat{v}_i(., t)$ and $\widehat{\Delta}_i(t)$.

Next definition provides a link between an elicitation process and the true value function – responses received from a truthful bidder should be properly captured by the elicitation process. In other words, the elicitation process ensures that a bidder who wishes to communicate its true value information does not have to respond to auctioneer's queries in untruthful manner. Such elicitation processes are refereed to as straightforward.

[8] To illustrate this definition, suppose that the auctioneer is limited to asking demand queries at linear prices in a setting with homogeneous items. Such elicitation process is not fully expressive. For example, suppose that $v(1) = 5$ and $v(2) = 20$. Then bidder's demand is either 2 when $p \leq 10$ or 0 when $p > 10$. Thus, this process does not allow to elicit $v(1)$.

Definition 2. *An elicitation process is called* ***straightforward*** *if it converts truthful responses of each bidder i made on the interval $[0, t]$ into a function $\hat{v}_i(., t)$ and set of bundles $\widehat{\Delta}_i(t)$ such that:*

$$\widehat{\Delta}_i(t) \subseteq \Delta_i(t), \tag{2.6}$$

where $\Delta_i(t)$ is determined by (2.1) - (2.2) for true value function $v_i(.)$.

The requirement (2.6) of a straightforward elicitation process allows the auctioneer to use $\widehat{\Delta}_i(t)$ instead of $\Delta_i(t)$ to validate condition (2.4) that guarantees efficiency. Also note that the marginal value of bundle z relative to any other bundle y is given by $\hat{v}_i(z, t) - \hat{v}_i(y, t)$ which equal to true marginal value $v_i(z) - v_i(y)$ when both bundles belong to $\widehat{\Delta}_i(t)$ (and to $\Delta_i(t)$ by inclusion (2.6)). In other words, set $\widehat{\Delta}_i(t)$ contains bundles for which true marginal values relative to each other has been successfully elicited.

An elicitation process that is guaranteed to identify efficient allocations in all relevant economies must be able to eventually elicit bidder i's value function. Full elicitation of $v_i(.)$ implies that $\Delta_i(t) = \Omega$. To guarantee that $\Delta_i(t)$ iteratively goes towards Ω, we restrict the elicitation process to weakly expand $\widehat{\Delta}_i(t)$ with time.

Definition 3. *An elicitation process is called* ***iterative*** *if the set $\widehat{\Delta}_i(t)$ for each bidder i is weakly increasing in time, i.e.*

$$\widehat{\Delta}_i(t') \subseteq \widehat{\Delta}_i(t) \qquad \textit{for all} \quad t' \leq t. \tag{2.7}$$

For a straightforward elicitation process, restriction (2.7) forces $\Delta_i(t)$ to weakly expand with time when bidder i responds truthfully. With an iterative elicitation process, the auctioneer progressively builds up the set of bundles that can be efficiently awarded to each bidder i while never excluding any bundles that have been already added to the set.

Now we define some monotonicity restrictions that can be imposed on the elicitation process. These restrictions are not required for our main results, but they are frequently utilized in practice to yield monotonic discovery of the auction outcome.

One of the most popular restrictions utilized by the auctioneers is a restriction on direction of adjustments to approximations of the value functions. Such restriction ensures that approximation of the value function monotonically approaches the true value function.

Definition 4. *An iterative elicitation process is called* ***ascending (descending)*** *if for each bidder i and any bundle $z \in \widehat{\Delta}_i(t)$, $\hat{v}_i(z, t)$ is nondecreasing (nonincreasing) in time.*

In other words, once bundle z is added to $\widehat{\Delta}_i(t)$ and becomes a feasible award for bidder i, the ascending (descending) elicitation process restricts the movement of the approximation value $\hat{v}_i(z,t)$ to nonnegative (nonpositive) adjustments such that the true value $v_i(z)$ is approached from below (from above). When $z \notin \widehat{\Delta}_i(t)$, restricting the direction of adjustment for $\hat{v}_i(z,t)$ is unnecessary and can be counterproductive.[9]

We say that economy $E(M)$ is *cleared at time* t if there exists a tentative allocation $x^*(M,t) = (x_1, ..., x_n)$ that solves (2.3) such that

$$x_i \in \widehat{\Delta}_i(t) \qquad \forall i \in M. \tag{2.8}$$

When condition (2.8) is not satisfied at time t, we say that economy $E(M)$ is uncleared at time t.

Another desirable form of monotonicity in dynamic auctions is ensuring that any economy that is already cleared stays cleared in future. An elicitation process that violates such restriction would be forcing bidders to reveal more information about their values than necessary, thus, going against the privacy preservation goal. The next restriction on the elicitation process guarantees such monotonicity.

Definition 5. *An iterative elicitation process is* ***monotonic*** *if for each bidder* i*, any bundle* $y \in \widehat{\Delta}_i(t')$*, any bundle* $z \in \Omega$ *and any time* $t > t'$*:*

$$\hat{v}_i(z,t) - \hat{v}_i(z,t') \leq \hat{v}_i(y,t) - \hat{v}_i(y,t'). \tag{2.9}$$

Intuitively, (2.9) ensures that value adjustments on $[t', t]$ for all bundles in $\widehat{\Delta}_i(t)$ are at least as high as for bundles that do not belong to $\widehat{\Delta}_i(t)$.

Proposition 3. *If an iterative elicitation process is monotonic, then economy* $E(M)$ *that is cleared at time* t' *stays cleared at any later time* $t > t'$*.*

2.3 Dynamic Vickrey Auction

In this section we define a general class of dynamic Vickrey auctions and show that they implement the Vickrey outcome as an ex post equilibrium. This is a standard result in the literature on dynamic implementation of the VCG mechanism.

The final ingredient for our construction is an adjustment rule that forces the auction to move towards clearing all relevant economies. A natural adjustment rule in our setting is to force bidders who prevent the clearing of a relevant economy to reveal more competitive values. At the same time, putting pressure

[9] For example, the ascending auction design described in Section 3 might require a negative adjustment to $\hat{v}_i(z,t)$ when bundle z is added to $\widehat{\Delta}_i(t)$.

on a specific set of bidders can get in conflict with auction simplicity. To facilitate greater generality, we do not put any restrictions on the set of bidders who can be asked to revise their values.

Adjustment Rule (for an iterative elicitation process): There exist $\epsilon > 0$ and $\lambda(\epsilon) \in (0, +\infty)$ such that for any time interval $[t', t]$ such that $t - t' \geq \lambda(\epsilon)$, there is bidder $i \in N$ for whom either

$$\hat{v}_i(z, t) - \hat{v}_i(z, t') \geq \epsilon \qquad \text{for all} \quad z \in \widehat{\Delta}_i(t'), \tag{2.10}$$

or

$$\hat{v}_i(z, t) - \hat{v}_i(z, t') \leq -\epsilon \qquad \text{for some} \quad z \notin \widehat{\Delta}_i(t') \tag{2.11}$$

The specified adjustment rule ensures that at least one of the bidders is forced to revise its values by a positive amount that is bounded away from zero. Revising values on $[t', t]$ can be done in two ways. First, bidder i can be asked to increase its values for bundles in $\widehat{\Delta}_i(t')$ (condition (2.10)). This adjustment is typically used for ascending auctions where bidders are asked to increase their bids for their current demand (like in the English auction). Alternatively, bidder i can be asked to reduce its values for at least one bundle not in $\widehat{\Delta}_i(t')$ (condition (2.11)). This adjustment is typically associated with descending auctions where implicit values for bundles are decreasing until bidders start demanding them (like in the Dutch auction).[10]

The adjustment rule is intentionally very permissive, effectively allowing for implementations that in reality would be considered "sealed-bid" (in terms of properties such as privacy preservation). In practice, the auctioneer should design an appropriate adjustment rule that, to the extent possible, preserves the advantages traditionally associated with dynamic auctions. Next we specify a general class of dynamic Vickrey auctions.

Dynamic Vickrey Auction: A dynamic Vickrey auction is an auction procedure with the following components:

(1) *(elicitation process)* The procedure uses a straightforward iterative elicitation process;

(2) *(adjustment rule)* The procedure is subject to the adjustment rule (2.10) – (2.11);

(3) *(closing rule)* The procedure terminates when all economies in $\{E(N), E(N_{-1}), ..., E(N_{-n})\}$ are cleared at the same time (denote T the termination time);

[10]Note that whenever $0 \in \widehat{\Delta}_i(t')$, condition (2.10) cannot be satisfied for bidder i since $\hat{v}_i(0, t) = 0$, and condition (2.11) has to be used instead.

(4) *(outcome rule)* The procedure awards bundle $x_i^*(N,T)$ to bidder i in exchange for a payment

$$p_i^V = \sum_{j \in N_{-i}} \left[\hat{v}_j(x_j^*(N_{-i},T)) - \hat{v}_j(x_j^*(N,T))\right]. \tag{2.12}$$

Theorem 1. *If each bidder i responds truthfully in accordance with $v_i(.)$, a dynamic Vickrey auction implements the Vickrey outcome.*

Proof. Given the adjustment rule and an iterative elicitation process, the procedure would eventually force each bidder to fully reveal its $v(.)$ at which point $\widehat{\Delta}_i(t) = \Omega$ for all $i \in N$ and the closing condition is trivially satisfied. Therefore, the procedure has to end in finite time T. Since the elicitation process is straightforward, $x^*(N,T)$ is efficient by Proposition 1 and $p_i^V = y_i^V$ by Proposition 2. □

A standard result in the literature on the dynamic implementation of the VCG mechanism is an ex post equilibrium, and the standard approach is to ensure that each bidder has to bid according to some value function at all times. Such restrictions are known as activity rules. Once the dynamic auction is augmented with an activity rule, by Theorem 1 and incentive compatibility of the VCG mechanism, any deviations from true value functions are unprofitable. To establish the same result in our context, we do not need a separately defined activity rule. As was mentioned before, our definition of an elicitation process (see Definition 1) indirectly includes such provisions – an elicitation process always generates a single-valued approximation function $\hat{v}_i(.,t)$ and set of bundles $\widehat{\Delta}_i(t)$.

Theorem 2. *Truthful responses by all bidders is an ex post equilibrium of a dynamic Vickrey auction.*

Proof. Suppose that bidder i deviates from truthful responses. First, the dynamic Vickrey auction would end in finite time given truthful responses of other bidders (e.g., bidder i cannot block the auction from closing). Second, denote $\tilde{v}_i(z) = \hat{v}_i(z,T)$. By construction, $\tilde{v}_i(.)$ is well-defined. Due to the straightforward elicitation process, $\widehat{\Delta}_j(T) \subseteq \Delta_j(T)$ for all bidders in N_{-i}, and $\widehat{\Delta}_i(T) \subseteq \tilde{\Delta}_i(T)$ where $\tilde{\Delta}_i(T)$ is the analog of $\Delta_i(T)$ defined for the $\tilde{v}_i(.)$. By Propositions 1 and 2, the auction outcome corresponds to a Vickrey outcome for the value profile $(\tilde{v}_i, v_{-i})$ which is weakly dominated for bidder i. □

Theorems 1 and 2 show that the problem of dynamic implementation of the VCG mechanism can be decomposed into specifying an appropriate elicitation process and then running a dynamic Vickrey auction based on the specified elicitation process using the standard rules (adjustment rule, closing rule, and outcome rule). Auctioneers can focus their attention solely on developing customized elicitation processes suitable for their particular applications instead of designing new dynamic Vickrey auctions. In the next section, we illustrate our main findings by developing a new elicitation process that is driven by a single linear, anonymous and nondecreasing price trajectory. To the best of our knowledge, it is the first dynamic implementation of the Vickrey outcome in the heterogeneous setting that is governed by a "simple" price path.

3 Efficient Ascending Auction

In this section, we construct an efficient ascending auction with a simple elicitation process that is driven by a single linear and anonymous price trajectory.

3.1 Elicitation Process

At time $t = 0$, the auctioneer initializes K clock prices, one for each good, at zero, i.e., $p(0) = 0$. At any time $t \geq 0$, the auctioneer quotes current clock prices $p(t)$ to bidders, and each bidder i replies with a single bundle $x_i(t) \in \Omega$ that is treated as bidder i's demand at price vector $p(t)$. The price trajectory $p(.)$ is assumed to be nondecreasing, continuous and piecewise linear on $[0, +\infty)$. The demand function $x_i(.)$ for each bidder i is assumed to be a right-continuous function that has at most a finite number of discontinuities on $[0, +\infty)$ (e.g., a piecewise constant function).[11]

It is well known that limiting the elicitation process to demand queries while using a single linear and anonymous price trajectory does not produce a fully expressive elicitation process,[12] so the auctioneer needs to ask for more information when the need arise.

With ascending clock prices, a truthful bidder would never demand a bundle that is a superset of a bundle that he demanded at some earlier time (by the law of demand). Since values for such bundles can no longer be elicited with any feasible demand query, they have to be elicited in some other way. For example, the auctioneer can ask a bidder to report its marginal value for such bundle relative to its current demand.

Formally, we define the set of *revealed* bundles $\widehat{\Delta}_i(t)$ as the set that includes any bundle $z \in \Omega$ that is a superset of bidder i's demand at price vector $p(t')$ for some $t' \in [0, t]$, i.e.:

$$\widehat{\Delta}_i(t) = \{z \in \Omega : \quad \exists t' \in [0, t] \quad \text{such that} \quad z \geq x_i(t')\}. \tag{3.1}$$

[11] This assumption is without loss of generality. It can be shown that a piecewise constant demand function exists for any function $v(.)$ satisfying assumptions (A1) - (A2) provided that the price path $p(.)$ is continuous and piecewise linear on $[0, +\infty)$.

[12] See Mishra and Parkes (2007).

Note that by construction, this set can only expand with time, e.g., $\widehat{\Delta}_i(t') \subseteq \widehat{\Delta}_i(t)$ for any $t' \leq t$ necessarily resulting in an iterative elicitation process.

By definition, the set of revealed bundles $\widehat{\Delta}_i(t)$ can expand only when bidder i switches its demand at time t. When this happens, bidder i is required to report its marginal values relative to $x_i(t)$ for all bundles that are newly added to the set $\widehat{\Delta}_i(t)$. For any newly added bundle z, denote $\widetilde{mv}_i(z)$ the reported marginal value of z relative to $x_i(t)$; and $t(z)$ the time when bundle z is added to $\widehat{\Delta}_i(t)$.

Now we construct the approximation of value function $\hat{v}_i(., t)$ to complement the set $\widehat{\Delta}_i(t)$. We say that bidder i *bids truthfully according to its value function* $v_i(.)$ at time $t \geq 0$ if:

(a) Bidder i truthfully reports its demand at $p(t)$, e.g.,

$$x_i(t) \in \arg\max_{z \in \Omega} \, [v_i(z) - p(t)\, z]; \text{ and} \tag{3.2}$$

(b) Bidder i truthfully reports its marginal value for all bundles that are added to the set of revealed bundles at time t, e.g., for any bundle z that is added to $\widehat{\Delta}_i(t)$ at time t,

$$\widetilde{mv}_i(z) = v_i(z) - v_i(x_i(t)) \tag{3.3}$$

Bidder i is said to *bid truthfully on* $[0, t]$ *according to its value function* $v_i(.)$ if he bids truthfully at all $t' \in [0, t]$.

For any bundle $z \in \widehat{\Delta}_i(t)$, denote its revealed marginal value relative to bidder i's current demand $x_i(t)$ as:

$$mv_i(z, t) = \begin{cases} \int_t^{t'} p(s) dx_i(s) & \text{if } \exists\, t' \in [0, t] : x_i(t') = z \\ \int_t^{t(z)} p(s) dx_i(s) + \widetilde{mv}_i(z) & \textit{otherwise} \end{cases} \tag{3.4}$$

Intuitively, the marginal value for bundle z is either recovered via revealed preferences if bundle z was previously demanded by bidder i, or its value was elicited via an additional query at time $t(z)$ when the bundle was added to $\widehat{\Delta}_i(t)$.

Lemma 1. *If bidder i bids truthfully according to $v_i(.)$ on $[0, t]$, then for all $z \in \widehat{\Delta}_i(t)$, the revealed marginal value of bundle z relative to bundle $x_i(t)$ is the true marginal value, i.e.,*

$$mv_i(z, t) = v_i(z) - v_i(x_i(t)) \tag{3.5}$$

Using the revealed marginal values, the auctioneer constructs the current approximation of the value function as follows:

$$\hat{v}_i(z,t) = \begin{cases} p(t)\,x_i(t) + mv_i(z,t) & \text{if} \quad z \in \widehat{\Delta}_i(t) \\ p(t)\,z & \text{if} \quad z \notin \widehat{\Delta}_i(t) \end{cases} \tag{3.6}$$

The formula (3.6) is interpreted as follows. For any revealed bundle $z \in \widehat{\Delta}_i(t)$, the formula simply uses the revealed marginal value between bundle z and the current demand $x_i(t)$ imputing the current clock price for current demand $x_i(t)$. For any non-revealed bundle $z \notin \widehat{\Delta}_i(t)$, the formula imputes the current clock price of bundle z. The latter part is needed to guarantee that the elicitation process is straightforward. The properties of the elicitation process are summarized by Proposition 4.

Proposition 4. *If bidder i bids truthfully according to $v_i(.)$ on $[0,t]$, then $\hat{v}_i(.,t)$ as defined in (3.6) and $\widehat{\Delta}_i(t)$ as defined in (3.1) constitute an elicitation process that is straightforward, iterative and ascending.*

In general, this elicitation process is not monotonic. We show how this elicitation process can be adjusted to yield monotonicity in addition to all other properties in Section 5.[13]

Dynamic auctions based on price clocks have rich traditions of using some notion of excess demand in adjusting clock prices and determining whether the auction reached its end (i.e., closing rules). These traditions go back to the famous "Walrasian auctioneer" device that was used to explain how a given economy converges to its competitive equilibrium. In the next section, we define an appropriate notion of excess demand for our auction and develop the price adjustment process.

3.2 Excess Demand and Clock Increments

Most dynamic auctions use some form of aggregate demand to inform bidders about the current level of competition in the auction. As a reminder, economy $E(M)$ is cleared at time t if there exists a tentative assignment $x^*(M,t) = (x_1^*, ..., x_n^*)$ that assigns each bidder a bundle from its corresponding sets of revealed bundles, i.e., $x_i^* \in \widehat{\Delta}_i(t)$ for all $i \in M$. But then the classical notions of aggregate and excess demand are inadequate in this framework.

The failure to clear economy $E(M)$ can be traced to bidders whose tentative winnings are not in their sets of revealed bundles. This suggest a possible way to construct excess demand that accounts for overdemanded items. When bidder i

[13] We intentionally do not pursue monotonicity here for the ease of exposition.

is assigned a bundle from $\widehat{\Delta}_i(t)$, bidder i does not prevent economy $E(M)$ from clearing, and so bidder i's contribution towards the excess demand is zero. In contrast, when bidder i is assigned a bundle outside of $\widehat{\Delta}_i(t)$, bidder i does prevent economy $E(M)$ from clearing, and its contribution towards excess demand should reflect its current demand $x_i(t)$ for items that are not present in x_i^*. The approach has two sources for potential multiplicity. First, there can be multiple tentative value-maximizing allocations, and second, bidders might implicitly demand multiple bundles at $p(t)$.

To handle multiplicity, we define $X^*(M,t)$ the set of all solutions to the winner-determination problem in (2.3), i.e.,

$$X^*(M,t) \;=\; \arg\max_{x \in X} \sum_{j \in M} \hat{v}_j(x_j, t) \tag{3.7}$$

and $D_i(t)$ the revealed demand correspondence of bidder i at time t, i.e.,

$$D_i(t) \;=\; \arg\max_{z \in \widehat{\Delta}_i(t)} \left[\, \hat{v}_i(z,t) - p(t)\, z \,\right] \tag{3.8}$$

Formally, the set of excess demands *(e.g. excess correspondence)* for economy $E(M)$ at time t is defined as:

$$Z(M,t) = \left\{ \sum_{j \in M} z_j \right\} \tag{3.9}$$

such that there exists a tentative value-maximizing allocation $x^* \in X^*(M,t)$ and for each bidder $i \in M$:

$$z_i = \begin{cases} 0 & \text{if} \quad x_i^* \in \widehat{\Delta}_i(t) \\ \max\{\, 0, d - x_i^* \,\} & \text{if} \quad x_i^* \notin \widehat{\Delta}_i(t) \end{cases} \tag{3.10}$$

where $d \in D_i(t)$ and $\max\{0, d - x_i^*\}$ is a component-wise maximum of two vectors.

The next proposition shows that the clearing of economy $E(M)$ at time t is equivalent to a natural clearing condition – "zero excess demand".

Proposition 5. *If all bidders bid truthfully according to their $v(.)$ on $[0, +\infty)$, economy $E(M)$ is cleared at time t if and only if $0 \in Z(M,t)$.*

Proof. If economy $E(M)$ is cleared at t, then $0 \in Z(M,t)$ by definition of $Z(M,t)$. For the converse, suppose that $0 \in Z(M,t)$, but $E(M)$ is not cleared at t. Then for any tentative assignment $x^*(M,t)$, there is at least one bidder $i \in M$ such that $x_i^* \notin \widehat{\Delta}_i(t)$. For such bidder i, there is at least one good for which demand $d \in D_i(t)$ strictly exceeds the tentative award x_i^* (otherwise, $x_i^* \geq d$ and $x_i^* \in \widehat{\Delta}_i(t)$). But then $0 \notin Z(M,t)$. □

When economy $E(M)$ is not cleared at time t, any excess demand vector $z \in Z(M,t)$ is positive for at least one good. Then the auctioneer can direct the auction process towards clearing economy $E(M)$ by picking one of the excess demand vectors from $Z(M,t)$ and increasing clock prices for goods with excess demand. While there are many possible ways to pick one of the excess demand vectors from $Z(M,t)$, the simplest one is to choose the one that minimizes the price value of the excess demand, i.e.:

$$z(M,t) \in \underset{z \in Z(M,t)}{\arg\min}\; p(t)\, z \tag{3.11}$$

This approach offers an advantage of targeting the smallest excess demand when measured by the price value (and it picks the $z(M,t) = 0$ when $0 \in Z(M,t)$ and $p(t) > 0$).

In order to implement the Vickrey outcome, the auctioneer needs to clear the main economy and all relevant marginal economies (e.g., Theorem 1). An important question for the auctioneer is to determine the order in which these economies are targeted. In well-behaved settings, it is natural for the marginal economies to clear before the main economy since competition is stronger when all bidders are included.[14] Ability to clear all marginal economies before the main economy is highly desirable property since the unnatural clearing order generally creates incentive problems in applications.

Here we make a simplifying assumption that the auctioneer simultaneously targets all relevant economies. Define a cumulative excess demand at time t as the maximum of excess demands across all relevant economies, i.e.,

$$Z^k(t) = \max_{M \in \{N, N_{-1}, \ldots, N_{-n}\}} z^k(M,t) \qquad \text{for all} \qquad k = 1, \ldots, K \tag{3.12}$$

where $z(M,t)$ is given by (3.11).

A naive price adjustment process based on excess demand (which in turn depends on individual demands) can cause a potential problem with infinitive price oscillations.[15] To completely avoid this problem and to provide a way to implement this design in practice, we adopt a price adjustment process that adjusts the speed of price clocks at discrete times.[16]

Formally, the auctioneer initializes all price clocks at zero $p(0) = 0$ and asks for initial reports to build $\widehat{\Delta}_i(0)$ for each bidder $i \in N$. Then, for any time $t \geq 0$,

[14] For example, a setting with homogeneous good and decreasing marginal values.

[15] See Gul and Stacchetti (2000) and Ausubel (2006).

[16] In practice, this approach is known as "intra-round bidding" and, to the best of our knowledge, is the only known way of implementing "continuous" price clocks.

the auctioneer sets the clock price for good $k \in \{1, ..., K\}$ as follows:

$$p^k(t) = \begin{cases} p^k(t') + \epsilon\,(t - t') & \text{if} \quad Z^k(t') > 0 \\ p^k(t') & \text{if} \quad Z^k(t') = 0 \end{cases} \tag{3.13}$$

where t' is the highest integer such that $t' < t$ and $\epsilon > 0$. Intuitively, the price adjustment described in (3.13) uses the excess demand of bidders at integer time t' to determine which clock prices to increase in the time interval $(t', t' + 1]$ and then increases them at a constant speed $\epsilon > 0$. The procedure always results in a nondecreasing continuous piecewise-linear price path $p(.)$.

3.3 Efficient Ascending Auction

In this section we complete our specification of the ascending auction and show that it implements the Vickrey outcome as an ex post equilibrium.

Ascending Auction: Ascending auction is an auction procedure with the following components:

(1) The auctioneer initializes clock prices at zero $p(0) = 0$.

(2) At each time $t \geq 0$, the auctioneer quotes clock prices $p(t)$ and asks each bidder i to report (1) its demand $x_i(t)$; and (2) its marginal value for any bundle that has been added to the set of revealed bundles $\widehat{\Delta}_i(t)$ at time t (see Section 3.1 for details);

(3) At each time $t \geq 0$ and each bidder i, the auctioneer constructs $\widehat{\Delta}_i(t)$ and $\hat{v}_i(, t)$ using formulas (3.1) and (3.6). The auctioneer calculates the excess demand $Z(t)$ according to formula (3.12);

(4) If at time t, $Z(t) \neq 0$, the clock prices are adjusted using the price-adjustment process (3.13) and the process goes to step (2). If $Z(t) = 0$, then the process terminates ($T := t$). Bidder i is awarded bundle $x_i^*(N, T)$ in exchange for a payment

$$p_i^V = \sum_{j \in N_{-i}} \left[\hat{v}_j(x_j^*(N_{-i}, T)) - \hat{v}_j(x_j^*(N, T)) \right]. \tag{3.14}$$

Theorem 3. *If each bidder i bids truthfully according to $v_i(.)$, the ascending auction implements the Vickrey outcome.*

Proof. By Proposition 4, the elicitation process is straightforward, iterative and ascending. For each economy $E(M)$ that is not cleared at integer time t, $Z^k(t) > 0$ if product k is overdemanded in at least one relevant economy, and its price will be increased by $\epsilon > 0$ on the time interval $(t, t+1]$. The increasing clock prices for overdemanded goods will cause an increase in $\hat{v}_i(z,t)$ for all bundles $z \in \widehat{\Delta}_i(t)$ for bidder i who demands overdemanded item at time t (satisfies the adjustment rule (2.10) on a sufficiently long time interval). Then the Vickrey outcome is implemented by Theorem 1. □

By Proposition 4, $\widehat{\Delta}_i(t)$ as defined in (3.1) and $\hat{v}_i(.,t)$ as defined in (3.6) constitute an elicitation process only when bidder i bids truthfully according to some value function $v_i(.)$. However, bidder i can easily place bids that are inconsistent with any value function when left unconstrained. In the next section, we specify appropriate activity rules that make sure that each bidder bids truthfully according to some value function.

3.4 Activity Rules for Ascending Auction

The activity rules specified in this section are based on *the Generalized Axiom of Revealed Preference (GARP)*, a well-known rationality concept. The most famous result in the GARP literature is the Afriat's Theorem due to Afriat (1967). The theorem links GARP and existence of a value function that rationalizes demand $x_i(.)$ given the price trajectory $p(.)$. Using our notation, the Afriat's theorem is stated as follows:

Afriat's Theorem (1967). *Given price path $p(.)$, bidder i's demand $x_i(.)$ is rationalized by some value function $v(.)$ if and only if her demand $x_i(.)$ satisfies GARP on $[0,T]$, i.e.:*

$$(GARP) \qquad p(s)[x_i(s) - x_i(t)] + \int_s^t p(u)dx_i(u) \ \leq \ 0 \qquad \forall\ t,s \in [0,T]$$

Ausubel and Baranov (2017) introduced a notion of GARP violation that is useful for our purposes. Suppose that bidder i's demand satisfies GARP on $[0,T]$. Then one could ask whether bidder i would have violated GARP on $[0,t]$ by bidding for bundle z at time t by verifying the sign of $gv_i(z,t)$ defined below:

$$gv_i(t,z) \ = \ \max_{s\in[0,t]} \left\{ p(s)[x_i(s) - z] + \int_s^t p(u)dx_i(u) + p(t)[z - x_i(t)] \right\} \qquad (3.15)$$

Intuitively, $gv_i(t,z)$ is the net amount that can be extracted from bidder i in a series of transactions that starts and ends with the same bundle z. By construction, $gv_i(t,z) \geq 0$ for all $z \in \Omega$ and all $t \in [0,T]$. If $gv_i(t,z) = 0$, then zero amount of money can be extracted from bidder i, i.e., bidder i is rational and bidding z at time t does not violate GARP. In contrast, when $gv_i(t,z) > 0$, bidder i has demonstrated irrationality, and bidding for bundle z at time t violates GARP. Furthermore, Ausubel and Baranov (2017) show that the size of GARP violation $gv_i(t,z)$ is useful for computing the upper bounds on value range of the underlying value function.

The first activity rule is concerned with bidders' demands. Here we use a standard GARP activity rule that forces bidder i to submit demands that can be rationalized by some value function. In addition, bidder i is not allowed to bid on any supersets of bundles that he demanded before (at weakly lower prices).

AR1: At time $t \geq 0$, bundle $x \in \Omega$ is unacceptable as demand of bidder i if

(a) either $gv_i(t,x) > 0$ (i.e., GARP violation); or

(b) $x \geq x_i(s)$ for some time $s < t$.

The second activity rule is concerned with reported marginal values for bundles that were never directly demanded. It restricts the marginal value reports of bidder i at the time when new bundles are added to its set of revealed bundles $\widehat{\Delta}_i(t)$. The marginal value for newly added bundle z is limited from above by the highest possible value that is consistent with bidder i never demanding z on the time interval $[0,t]$.

AR2: For a bundle z that is added to $\widehat{\Delta}_i(t)$ at time t, the reported marginal value $\widetilde{mv}_i(z)$ has to satisfy the following inequality:

$$\widetilde{mv}_i(z) \;\leq\; p(t)\,[z - x_i(t)] - gv_i(t,z) \tag{3.16}$$

Next proposition shows that each bidder is forced to bid according to some value function when its bidding is restricted by activity rules AR1 and AR2.

Proposition 6. *Given nondecreasing price path $p(.)$, bidder i bids truthfully according to some value function $\tilde{v}_i(.)$ on $[0,T]$ if and only if its bidding is constrained by activity rules AR1 and AR2.*

The most interesting part of Proposition 6 is that constraining reported marginal values with AR2 at time t is sufficient for existence of a value function that rationalizes bidding even at later times $t' \geq t$. In other words, the upper bound for $\widetilde{mv}_i(z)$ from (3.16) cannot decrease at later time. As a result, any marginal value allowed at time t stays consistent with some truthful bidding at a later time (assuming nondecreasing clock price trajectory).

Finally, Theorem 3 and Proposition 6 together imply that truthful bidding by all bidders is an ex post equilibrium of an ascending auction.

Theorem 4. *Truthful responses by all bidders is an ex post equilibrium of an ascending auction with activity rules AR1 and AR2.*

4 Illustrative Example of Ascending Auction

To illustrate the efficient ascending auction, we use a classic LLG - type example with two items, A and B, and three bidders.[17]. Bidder 1 is a local bidder who values item A at 3 and has no value for item B. Bidder 2 is another local bidder who values item B at 10 and has no value for item A. Finally, Bidder 3 is a global bidder who values a combination of items A and B at 8, and her value for a standalone item, either A or B, is only 2.

Suppose that the auctioneer starts the auction with $p(0) = (0,0)$ and then uses the following adjustment process for the clock price of good $k \in \{A, B\}$:

$$\dot{p}^k(t) = \begin{cases} 1 & \text{if} \quad Z^k(t) > 0 \\ 0 & \text{if} \quad Z^k(t) = 0 \end{cases}$$

We assume that bidders demand the smallest bundle when indifferent. The detailed history for this example that includes approximations of value functions is provided in Table 1.

The price path develops in three stages. In stage one, both prices rise from $p(0) = (0,0)$ to $p(3) = (3,3)$. At $p(3) = (3,3)$, Bidder 1 reduces its demand to $(0,0)$. At this time, the auctioneer knows that value of Bidder 1 for A and AB is 3, but does not know its value for item B. The auctioneer asks Bidder 1 to report its marginal value for item B relative to the null bundle (current demand of Bidder 1), and the value function of Bidder 1 is fully revealed.

In stage two, clock prices rise from $p(3) = (3,3)$ to $p(5) = (3,5)$. At $p(5) = (3,5)$, Bidder 3 reduces its demand to $(0,0)$. At this time, the auctioneer knows that value of Bidder 3 for AB is 8, but does not know its value for items A or B. The auctioneer asks Bidder 3 to report its marginal value for both item A and B relative to the null bundle (current demand of Bidder 3), and the value function of Bidder 3 is fully revealed. There are three observation that should ba made at this moment. First, the regular Walrasian demand at $p(5) = (3,5)$ is only (0,1), below supply. The excess demand for the main economy is $Z(N,5) = (0,0)$ indicating that the main economy is cleared. At the same time, the excess demand that also accounts for demand in marginal economies is still at $Z(5) = (0,1)$ since the marginal economy for Bidder 1, $E(N_{-1})$, has not been cleared yet. The other marginal economies $E(N_{-2})$ and $E(N_{-3})$ are also cleared at $t = 5$.

In the last stage, clock prices rise from $p(5) = (3,5)$ to $p(7) = (3,7)$. At $p(7) = (3,7)$, the marginal economy $E(N_{-1})$ is finally cleared, and excess demand equals zero. Bidder 1 is awarded item A and charged $\hat{v}_3(AB) - \hat{v}_2(B) = 1$. Bidder 2 gets item B and pays $\hat{v}_3(AB) - \hat{v}_1(A) = 5$. Note that while values of Bidder 1

[17]LLG stands for the Local Local Global setting first introduced by Krishna and Rosental (1996)

Table 1: An Illustrative Example of the Ascending Auction

	Bidder 1	*Bidder 2*	*Bidder 3*		
Values (A, B, AB):	$v_1 = (3, 0, 3)$	$v_2 = (0, 10, 10)$	$v_3 = (1, 1, 8)$		
Efficient Assignment:	A	B	$\emptyset$		
Vickrey Payments:	1	5	0		
Clock Price	Demand/Value Approximations			$Z(t)$	$Z(N, t)$
$p(t) = (t, t)$ $t : 0 \to 3$	$x_1 = (1, 0)$ $\hat{v}_1 = (t, t, t)$ $\widehat{\Delta}_1 = \{A, AB\}$	$x_2 = (0, 1)$ $\hat{v}_2 = (t, t, t)$ $\widehat{\Delta}_2 = \{B, AB\}$	$x_3 = (1, 1)$ $\hat{v}_3 = (t, t, 2t)$ $\widehat{\Delta}_3 = \{AB\}$	(1, 1)	(1, 1)
$p(t) = (3, t)$ $t : 3 \to 5$	$x_1 = (1, 0)$ $\hat{v}_1 = (3, 0, 3)$ $\widehat{\Delta}_1 = \Omega$	$x_2 = (0, 1)$ $\hat{v}_2 = (3, t, t)$ $\widehat{\Delta}_2 = \{B, AB\}$	$x_3 = (1, 1)$ $\hat{v}_3 = (3, t, 3 + t)$ $\widehat{\Delta}_3 = \{AB\}$	(0, 1)	(0, 1)
$p(t) = (3, t)$ $t : 5 \to 7$	$x_1 = (1, 0)$ $\hat{v}_1 = (3, 0, 3)$ $\widehat{\Delta}_1 = \Omega$	$x_2 = (0, 1)$ $\hat{v}_2 = (3, t, t)$ $\widehat{\Delta}_2 = \{B, AB\}$	$x_3 = (0, 0)$ $\hat{v}_3 = (1, 1, 8)$ $\widehat{\Delta}_3 = \Omega$	(0, 1)	(0, 0)
$p(7) = (3, 7)$	$x_1 = (0, 0)$ $\hat{v}_1 = (3, 0, 3)$ $\widehat{\Delta}_1 = \Omega$	$x_2 = (0, 1)$ $\hat{v}_2 = (3, 7, 7)$ $\widehat{\Delta}_2 = \{B, AB\}$	$x_3 = (0, 0)$ $\hat{v}_3 = (1, 1, 8)$ $\widehat{\Delta}_3 = \Omega$	(0, 0)	(0, 0)

and Bidder 3 are fully elicited in this example, Bidder 2 reveled his value function only partially.

5 Additional Discussion

In this section, we provide discussion of several issues related to implementation of the ascending auction described in Section 3.

5.1 Monotonic Elicitation Process

In general, the elicitation process defined by (3.1) and (3.6) is not monotonic.[18] This is undesirable since an economy that was cleared at time t' can be noncleared at a later time t, which in turn would lead to excessive elicitation. Formula

[18]It is monotonic in a homogeneous setting with multiple units, i.e., $K = 1$.

(5.1) below provides an alternative to formula (3.6) that guarantees monotonicity of the elicitation process (e.g., Definition 5). As shown in Proposition 7, the GARP violation term from section 3.4 can be used to eliminate the source of nonmonotonicity without sacrificing other properties.

$$\hat{v}_i(z,t) = \begin{cases} p(t)\, x_i(t) + mv_i(z,t) & \text{if} \quad z \in \widehat{\Delta}_i(t) \\ p(t)\, z - gv_i(t,z) & \text{if} \quad z \notin \widehat{\Delta}_i(t) \end{cases} \tag{5.1}$$

Proposition 7. *If bidder i bids truthfully according to $v_i(.)$ on $[0,t]$, then $\hat{v}_i(.,t)$ as defined in (5.1) and $\widehat{\Delta}_i(t)$ as defined in (3.1) constitute an elicitation process that is straightforward, iterative, ascending and monotonic.*

5.2 Dynamic Feedback

One of the most desirable properties of the dynamic auctions is their ability to progressively update bidders about their prospective winnings and payments. In our auction, current bidder demand and current clock prices can be very imprecise approximations of the winning bundles and associated payments, and the auctioneer can aid bidders by providing them with more relevant estimates.

For example, at each time t, the auctioneer can inform bidder i about its current tentative assignment $x_i^*(N,t)$ for the main economy, and its current payment calculated using (3.14) evaluated at time t. For bidder i who is tentatively assigned bundle $x_i^*(N,t) \in \widehat{\Delta}_i(t)$, such feedback is adequate since bidder i indeed submitted the bid for this bundle at some point. In contrast, when bidder i is assigned bundle $x_i^*(N,t) \notin \widehat{\Delta}_i(t)$, such feedback can get very frustrating.

A better approach is to provide each bidder i with its allocation in a constrained WDP where bidder i is guaranteed to win one of its revealed bundles from $\widehat{\Delta}_i(t)$. The corresponding Vickrey payment is calculated using the regular counterfactual WDP in which bider i is assigned the null bundle.

5.3 Reducing Informational Burden

When the number of possible bundles is large, the elicitation process proposed in Section 3 can ask bidders to report their marginal values for a large number of bundles at a time. In practice, the auctioneer should ask bidders to submit marginal values only for bundles of their interest that can potentially win and assume zero incremental value for any bundle that does not receive a bid. In addition, the clock nature of our auction allows the auctioneer to reduce the informational burden on bidders by dynamically eliminating bundles that has been proven to be irrelevant.

Proposition 1 in Section 2 shows that a tentative allocation $x^*(M,t) = (x_1, ..., x_n)$ is efficient for economy $E(M)$ if $x_i \in \Delta_i(t)$ for all bidders in M. In a symmetric way, we can establish inefficiency of a feasible allocation $x \in X$.

Proposition 8. *A feasible allocation $x = (x_1, ..., x_n)$ is inefficient for economy $E(M)$ if there exists a feasible allocation $y = (y_1, ..., y_n)$ such that $y_i \in \Delta_i(t)$ for all $i \in M$ and*

$$\sum_{j \in M} \hat{v}_j(x_j, t) < \sum_{j \in M} \hat{v}_j(y_j, t). \tag{5.2}$$

By Proposition 8, a bundle is irrelevant for a given bidder at time t if it can be established that assigning this bundle to a bidder leads to an inefficient allocation in all relevant economies. Proposition 8 provides an actionable way to test bundles for irrelevance during the auction process by replacing $\Delta_i(t)$ with $\widehat{\Delta}_i(t)$. The complete description of such test can be found in Appendix B. In case bundle z is proven to be irrelevant at a time when its marginal value should be elicited, the auctioneer informs the bidder that there is no need for that.

6 Conclusion

Demand for dynamic implementations of the VCG mechanism in various settings is historically strong. The focus of the economic literature on standard elicitation practice (quoting prices and asking for bidder's demand at these prices) can result in very impractical auction designs. We show that introducing flexible elicitation tools facilitates auction designs that can be more appealing to practitioners.

In this paper, we develop a general framework for a dynamic Vickrey auction and show that the task of designing an efficient dynamic auction can be effectively reduced to a task of designing an elicitation process that satisfies a few properties. This decomposition allows the auctioneer to create a dynamic Vickrey auction based on any elicitation process subject to several properties. This is highly desirable since it allows the auctioneer to tailor the specifics of the elicitation process to the needs of a particular application without worrying about other design elements.

To illustrate our result, we design an efficient ascending auction that uses an elicitation process based on a linear, anonymous and ascending price trajectory. To the best of our knowledge, it is a first dynamic auction that uses simple and practical elicitation tool to implement the Vickrey outcome in a general private value setting. The key idea for this auction is to use the standard clock price - demand queries until this protocol does not allow a bidder to communicate value information that can be important. At that movement, the auctioneer recovers missing information via additional direct value queries.

Our ascending auction is specified for a continuous price trajectory. It is possible to construct a similar auction design that operates on the discrete price trajectory. This “discrete” design requires minor changes in the elicitation process and activity rules (e.g., a discrete version of the GARP activity rule).

Finally, the practical appeal of our ascending auction depends on the specifics of the auction setting. The design is especially attractive in settings with a relatively small number of bundles available for bidding. For example, it appears to be a good fit for the upcoming PSSR spectrum auction in UK in which Ofcom plans to allocate 4 lots in the 2.3 GHz band and 30 lots in the 3.4 GHz band. At the same time, the product setting (forward part) of the Incentive auction held in US in 2016 - 2017 appears to be a bad fit for our design due to extremely large number of heterogeneous goods (regional licenses at 416 partial economic areas).

A Appendix - Proofs

PROOF OF PROPOSITION 3. Suppose that $x^*(M,t') = (x_1^*, ..., x_n^*)$ clears economy $E(M)$ at time t', i.e., $x_j^* \in \widehat{\Delta}_j(t')$ for all $j \in M$ and

$$\sum_{j \in M} \hat{v}_j(x_j^*, t') \geq \sum_{j \in M} \hat{v}_j(x_j, t') \qquad \forall x \in X$$

By Definition 5, for any time $t > t'$

$$\sum_{j \in M} \hat{v}_j(x_j^*, t) \geq \sum_{j \in M} \hat{v}_j(x_j, t) \qquad \forall x \in X$$

since $x_j^* \in \widehat{\Delta}_j(t')$ for all $j \in M$. Since the elicitation process is iterative, $x_j^* \in \widehat{\Delta}_j(t)$ for all $j \in M$ and economy $E(M)$ is cleared at time t. □

PROOF OF LEMMA 1. First, suppose that there exists $t' \in [0, t)$ such that $x_i(t') = z$. Denote $t_1, ..., t_m$ all times in the interval $[t', t]$ when bidder i switched its demand. Given the continuous price path $p(.)$, bidder i who bid truthfully is indifferent at all switch points, i.e.,

$$\begin{array}{rcl} v_i(x_i(t')) - p(t_1)\, x_i(t') & = & v_i(x_i(t_1)) - p(t_1)\, x_i(t_1) \\ v_i(x_i(t_1)) - p(t_2)\, x_i(t_1) & = & v_i(x_i(t_2)) - p(t_2)\, x_i(t_2) \\ ... & ... & ... \\ v_i(x_i(t_{m-1})) - p(t_m)\, x_i(t_{m-1}) & = & v_i(x_i(t)) - p(t_m)\, x_i(t) \end{array}$$

But then

$$\begin{aligned} v_i(x_i(t')) &= v_i(x_i(t)) - p(t_1)[x_i(t_1) - x_i(t')] + ... + p(t_m)[x_i(t) - x_i(t_{m-1})] \\ &= v_i(x_i(t)) - \textstyle\int_{t'}^{t} p(u) dx_i(u) \end{aligned}$$

Given that $x_i(t') = z$ and (3.4), we have:

$$mv_i(z, t) = \int_t^{t'} p(u) dx_i(u) = v_i(z) - v_i(x_i(t))$$

Second, if $z \in \widehat{\Delta}_i(t)$, but it was never explicitly demanded by bidder i, then by (3.4) and definition of truthful bidding, we have:

$$\begin{aligned} mv_i(z, t) &= \textstyle\int_t^{t(z)} p(u) dx_i(u) + \widetilde{mv}_i(z) \\ &= v_i(x_i(t(z))) - v_i(x_i(t)) + v_i(z) - v_i(x_i(t(z))) \\ &= v_i(z) - v_i(x_i(t)) \end{aligned}$$

□

PROOF OF PROPOSITION 4. First, the formulas in (3.1) and (3.6) are indeed represent an elicitation process since they uniquely define $\widehat{\Delta}_i(t)$ and $\hat{v}_i(.,t)$ at each time t and each bidder i. By construction, this elicitation process is iterative and ascending (since $p(.)$ is nondecreasing).

Straightforward: Since bidder i bids truthfully according to $v(.)$ on $[0,t]$, then for any bundle $z \in \widehat{\Delta}_i(t)$ (due to Lemma 1)

$$\begin{aligned} \hat{v}_i(z,t) &= p(t)\,x_i(t) + mv_i(z,t) \\ v_i(z) - \hat{v}_i(z,t) &= v_i(x_i(t)) - p(t)\,x_i(t) \\ \delta_i(z,t) &= \delta_i(x_i(t),t) \end{aligned}$$

For any bundle $z \notin \widehat{\Delta}_i(t)$, given that bidder i bids truthfully at $p(t)$,

$$\begin{aligned} v_i(z) - p(t)\,z &\leq v_i(x_i(t)) - p(t)\,x_i(t) \\ \delta_i(z,t) &\leq \delta_i(x_i(t),t) \end{aligned}$$

Therefore, any bundle z from $\widehat{\Delta}_i(t)$ also belongs to $\Delta_i(t)$.

$\square$

PROOF OF PROPOSITION 6. (Necessity) If bidder i bids according to $\tilde{v}_i(.)$, then bidder i never violates AR1(a) by Afriat's theorem. Note that AR1(b) never gets in a way of truthful bidding since if $gv_i(t,x) = 0$ and $x \geq x_i(s)$ for some time $s < t$, then it must be the case that both x and $x_i(s)$ are both optimal demands at time t, i.e.,

$$\{x, x_i(s)\} \in \arg\max_{z \in \Omega}\, [\tilde{v}_i(z) - p(t)\,z].$$

For AR2, by revealed preference,

$$\begin{aligned} \tilde{v}_i(z) &\leq \min_{s\in[0,t]} \{\tilde{v}_i(x_i(s)) + p(s)[z - x_i(s)]\} \\ &= \min_{s\in[0,t]} \left\{\tilde{v}_i(x_i(t)) - \textstyle\int_s^t p(u)dx_i(u) + p(s)[z - x_i(s)]\right\} \\ &= \tilde{v}_i(x_i(t)) + \min_{s\in[0,t]} \left\{-p(s)[x_i(s) - z] - \textstyle\int_s^t p(u)dx_i(u)\right\} \\ &= \tilde{v}_i(x_i(t)) + p(t)[z - x_i(t)] - gv_i(t,z) \end{aligned}$$

and the marginal value between bundles z and $x_i(t)$ is

$$\tilde{v}_i(z) - \tilde{v}_i(x_i(t)) \leq p(t)[z - x_i(t)] - gv_i(t,z) = \widetilde{mv}_i(z)$$

Thus, AR2 never overconstrains a bidder who bids according to some value function $\tilde{v}_i(.)$.

(Sufficiency) Suppose that AR1 and AR2 hold on $[0, T]$. Construct $\tilde{v}_i(.)$ as

$$\tilde{v}_i(z) = \begin{cases} p(T)\, x_i(T) - \int_{t(z)}^{T} p(u) dx_i(u) + \widetilde{mv}_i(z) & z \in \widehat{\Delta}_i(T) \\ 0 & z \notin \widehat{\Delta}_i(T) \end{cases}$$

where $t(z) = t$ and $\widetilde{mv}_i(z) = 0$ if bundle z was demanded by bidder i at time t. To show that $\tilde{v}_i(.)$ rationalizes bidding of bidder i at any time $s \in [0, T]$:

1. For bundle z that was demanded at some time $t \in [0, T]$ (i.e., $x_i(t) = z$), the bidding is rationalized since:

$$\tilde{v}_i(z) - p(s)\, z \leq \tilde{v}_i(x_i(s)) - p(s)\, x_i(s)$$

$$p(s)\,[x_i(s) - z] - \int_t^T p(u) dx_i(u) + \int_s^T p(u) dx_i(u) \leq 0$$

$$p(s)[x_i(s) - z] + \int_s^t p(u) dx_i(u) \leq 0$$

The last inequality is satisfied due to GARP verification.

2. For bundle $z \in \widehat{\Delta}_i(T)$ that was never demanded explicitly but was added to the set of revealed bundles at time $t(z)$, the bidding is rationalized since:

   - For $s \leq t(z)$, the bidding rationalized by AR2 through the following inequalities:

$$\tilde{v}_i(z) - p(s)\, z \leq \tilde{v}_i(x_i(s)) - p(s)\, x_i(s)$$

$$p(s)\,[x_i(s) - z] - \int_{t(z)}^{T} p(u) dx_i(u) + \widetilde{mv}_i(z) + \int_s^T p(u) dx_i(u) \leq 0$$

$$\widetilde{mv}_i(z) \leq -p(s)[x_i(s) - z] - \int_s^{t(z)} p(u) dx_i(u)$$

$$\widetilde{mv}_i(z) \leq p(t(z))[z - x_i(t(z))] - gv_i(t(z), z)$$

   - For $s > t(z)$, we show that $gv_i(t(z), z) = gv_i(s, z)$ (GARP violation for bundle z is not going to increase over time interval $[t(z), s]$). Then the bidding is rationalized by the same argument as for the case $s \leq t(z)$. Suppose that there exist $s' \in (t(z), s]$ such that garp violation is strictly higher for s' than for $t(z)$ such that $z \geq x_i(t(z))$ and $z \neq x_i(t(z))$ ($x_i(t(z)) \leq z$ since bundle z is added to the set of revealed bundles at time $t(z)$). Then

$$p(s')[x_i(s') - z] + \int_{s'}^{s} p(u) dx_i(u) > p(t(z))[x_i(t(z)) - z] + \int_{t(z)}^{s} p(u) dx_i(u)$$

$$p(s')[x_i(s') - x_i(t(z))] + \int_{s'}^{t(z)} p(u) dx_i(u) > [p(s') - p(t(z))][z - x_i(t(z))] \geq 0$$

   The last inequality implies the GARP violation at time s'. Hence, s' cannot maximize (3.15) and $gv_i(t(z), z) = gv_i(s, z)$.

□

PROOF OF PROPOSITION 7. First, the formulas in (3.1) and (5.1) are indeed represent an elicitation process since they uniquely define $\widehat{\Delta}_i(t)$ and $\hat{v}_i(.,t)$ at each time t and each bidder i. By construction, this elicitation process is iterative and ascending (since $p(.)$ is nondecreasing).

Straightforward: Since bidder i bids truthfully according to $v(.)$ on $[0,t]$, then for any bundle $z \in \widehat{\Delta}_i(t)$ (due to Lemma 1)

$$\begin{aligned} \hat{v}_i(z,t) &= p(t)\,x_i(t) + mv_i(z,t) \\ v_i(z) - \hat{v}_i(z,t) &= v_i(x_i(t)) - p(t)\,x_i(t) \\ \delta_i(z,t) &= \delta_i(x_i(t),t) \end{aligned}$$

For any bundle $z \notin \widehat{\Delta}_i(t)$, given that bidder i bids truthfully on $[0,t]$,

$$\begin{aligned} v_i(z) &\leq \min_{s\in[0,t]} \{v_i(x_i(s)) - p(s)\,[x_i(s) - z]\} \\ &= v_i(x_i(t)) + \min_{s\in[0,t]} \{ - \textstyle\int_s^t p(u)dx_i(u) - p(s)\,[x_i(s) - z]\} \\ &= v_i(x_i(t)) + p(t)\,[z - x_i(t)] - gv_i(t,z) \end{aligned}$$

But then

$$\begin{aligned} v_i(z) - [p(t)\,z - gv_i(t,z)] &\leq v_i(x_i(t)) - p(t)\,x_i(t) \\ \delta_i(z,t) &\leq \delta_i(x_i(t),t) \end{aligned}$$

Therefore, any bundle z from $\widehat{\Delta}_i(t)$ also belongs to $\Delta_i(t)$.

Monotonicity: For any bundle $z \notin \widehat{\Delta}_i(t)$, and $t' < t$

$$\begin{aligned} \hat{v}_i(z,t) - \hat{v}_i(z,t') &= [p(t) - p(t')]\,z - [gv_i(t,z) - gv_i(t',z)] \\ &\leq [p(t) - p(t')]\,z - [\textstyle\int_{t'}^t p(u)dx_i(u) + p(t)\,[z - x_i(t)] - p(t')\,[z - x_i(t')]] \\ &= p(t)\,x_i(t) - \textstyle\int_{t'}^t p(u)dx_i(u) - p(t')\,x_i(t') \\ &= \hat{v}_i(x_i(t'),t) - \hat{v}_i(x_i(t'),t') \\ &= \hat{v}_i(y,t) - \hat{v}_i(y,t') \qquad \forall y \in \widehat{\Delta}_i(t') \end{aligned}$$

□

PROOF OF PROPOSITION 8.

$$\begin{aligned} \sum_{j\in M} v_j(x_j) &= \sum_{j\in M} [\hat{v}_j(x_j,t) + \delta_j(x_j,t)] \\ &< \sum_{j\in M} [\hat{v}_j(y_j,t) + \delta_j(y_j,t)] \\ &= \sum_{j\in M} v_j(y_j) \end{aligned}$$

Since both x and y are feasible allocations, x has to be inefficient. □

B Appendix - Test for Irrelevant Bundles

For economy $E(M)$ and bidder $i \in M$, the following steps have to be carried out to test bundle z for irrelevance at time t:

1. Find the maximal value that can be obtained by giving each bidder in M a bundle from its set of revealed bundles $\widehat{\Delta}_i(t)$, i.e., solve the following problem:

$$\begin{aligned} H1 \;=\; & \max_{y \in X} \sum_{j \in M} \hat{v}_j(y_j, t) \\ \text{s.t.} \quad & y_j \in \widehat{\Delta}_j(t) \quad \forall j \in M \end{aligned} \tag{B.1}$$

   If the solution for this problem does not exist, the test cannot proceed to step 2.

2. Find the maximal value that can be obtained by allocating bundle z to bidder i, i.e., solve the following problem:

$$\begin{aligned} H2 \;=\; & \max_{x \in X} \sum_{j \in M/\{i\}} \hat{v}_j(x_j, t) \\ \text{s.t.} \quad & x_i = z \end{aligned} \tag{B.2}$$

3. Bundle z is irrelevant if

$$p(t)\, z - gv_i(z, t) < H1 - H2. \tag{B.3}$$

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