The VCG Mechanism, the Core, and Assignment Stages in Auctions

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Abstract

The Vickrey-Clarke-Groves (VCG) mechanism is one of the most compelling constructs in mechanism design, but complementarities create the possibility of non-core and even zero-revenue outcomes. In this article, we develop a theory of complementarities in one of the most natural applications for the VCG mechanism: assignment stages of spectrum auctions. When two opponents bid on distinct parts of a bidder's preferred assignment, they set up a direct bidder complementarity. In addition, the constraint in assignment stages that guarantees contiguous spectrum to each participant is shown to enable a new, indirect complementarity. As a result, non-core outcomes, together with all of the accompanying anomalies, must be expected in VCG assignment stages. We then examine a conspicuous recent spectrum auction, the FCC's Broadcast Incentive Auction, and we provide the first empirical documentation of zero-revenue outcomes.

Keywords: VCG mechanism, Vickrey auction, core, spectrum auction, assignment stage

JEL codes: D44, D47

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1 Introduction

One of the most elegant and compelling constructs in the toolbox of mechanism design is the Vickrey-Clarke-Groves (VCG) mechanism.[1] In appropriate economic environments, asking agents to report their valuations for goods, allocating the goods efficiently relative to these reports, and compensating agents according to an opportunity cost calculation relative to these reports gives rise to an elegant mechanism. Truth-telling becomes a dominant strategy inducing efficiency.

However, two decades ago, economists added a cautionary amendment to our understanding: complementarities of goods can induce complementarities of agents, leading to potential anomalies in the VCG mechanism. The auction may result in uncompetitively low revenues (i.e., VCG payments that lie outside the core, in the sense that there exists a coalition of losing bidders who offered to pay more than the winning prices for some of the goods), and may invite perversities such as "loser collusion".[2]

A zero-revenue outcome, the most extreme ramification of this problem, becomes a theoretical possibility. Consider the "local-local-global" (LLG) model, with two goods (A and B) and three bidders (1, 2 and 3) possessing the following valuations:[3]

$$\begin{array}{llll} v_1(AB) = 2 & v_1(A) = 2 & v_1(B) = 0 & v_1(\emptyset) = 0 \\ v_2(AB) = 2 & v_2(A) = 0 & v_2(B) = 2 & v_2(\emptyset) = 0 \\ v_3(AB) = 2 & v_3(A) = 0 & v_3(B) = 0 & v_3(\emptyset) = 0 \end{array}$$

Local bidder 1 values only good A, at 2; and local bidder 2 values only good B, also at 2. The global bidder 3 views goods A and B as perfect complements and values only the AB combination at 2. The VCG mechanism efficiently allocates good A to bidder 1 and good B to bidder 2. However, each local bidder pays zero, as neither is individually "responsible" for displacing the global bidder, who has complementary preferences for the two goods. Not only are the zero revenues disappointing, but the VCG outcome lies outside of the core, since there is a blocking coalition (the seller and the global bidder) whose coalitional value equals 2.

Low revenues by themselves are not necessarily a major concern for some sellers, especially government agencies that prioritize efficiency. At the same time, the possibility of zero-revenue outcomes permits bidders to obtain valuable items for free, a sure recipe for exploiting the auction mechanism.

Despite the theoretical literature, we are unaware of any documented empirical examples of zero-revenue outcomes. One possible reason for the dearth of known examples is that these observations have spawned a literature on core-selecting mechanisms[4]—and real-world auctioneers, aware of the

[1] Vickrey (1961), Clarke (1971), Groves (1973).
[2] Ausubel and Milgrom (2002, 2006).
[3] Ausubel and Milgrom (2002) at p. 5.
[4] Day and Raghavan (2007), Day and Milgrom (2008), Day and Cramton (2012).

problems, frequently include core adjustments in their rules. An alternative explanation for the lack of examples is that zero-revenue outcomes rely upon extreme and unrealistic complementarities of goods, rendering it merely a theoretical curiosity and not a practical problem.

In this article, we document the occurrence of zero-revenue and other non-core outcomes in what may appear at first glance to be an unlikely place: the assignment stage of a spectrum auction. Auctions of spectrum licenses that are close substitutes are often dichotomized into two successive phases. First, in the allocation stage, bidders bid for quantities of "generic" licenses (spectrum blocks that are not yet assigned to any specific frequency). Second, in the assignment stage, bidders compete for different assignments of specific frequencies in the quantity of their generic winnings from the allocation stage.

Before considering the issues discussed herein, the VCG mechanism may seem to be the most natural methodology for an assignment stage. With the bulk of the auction value already determined in the first stage, VCG provides a fast (single round, sealed bid) and simple (dominant strategy) way to elicit bidders' preferences for specific frequencies and to establish the value-maximizing assignment. What could possibly go wrong? The answer offered by this article is that bidder complementarities—and, hence, non-core outcomes—are endemic to assignment stages. As such, all pathologies possible in the VCG mechanism are especially likely to materialize in this setting.

It is initially easy to overlook that complementarities would even be present in assignment stages of spectrum auctions. The complementarities that are typically associated with the telecom industry relate to economies of scale within a geographic region or to synergies between regions. But once the allocation stage has concluded and the number of spectrum blocks won by each bidder in each region has been determined, any of these complementarities should have been resolved. Another source of complementarities is bidders' preferences for contiguous spectrum. For example, in an auction of four spectrum blocks that are identified (in ascending order of frequencies) as the A, B, C and D blocks and are otherwise equivalent, a winner of two blocks in the allocation stage would much prefer to be assigned the AB combination, the BC combination, or the CD combination. However, spectrum regulators are fully cognizant of this preference and impose a "contiguity restriction" assuring that all multi-block winners receive adjacent blocks. One might naïvely hope that this restriction would treat all residual complementarities.

We show that this hope is misplaced. While imposing the contiguity restriction does mitigate certain complementarities among bidders, it simultaneously enables a novel type of bidder complementarity. More precisely, an assignment stage that respects contiguity is subject to two different types of complementarities. First, a bidder is deprived of a desired assignment if each of two opponents demands distinct frequencies overlapping the desired assignment (a *direct* complementarity). Second, the bidder is also deprived of the desired assignment if opponents demand frequencies which, while not overlapping, each render it infeasible for the bidder to win the desired assignment via the enforcement of contiguity (an *indirect* complementarity). In each scenario, neither complementary opponent would be individually responsible for depriving the bidder of its desired assignment, facilitating non-core outcomes. Note that, without the contiguity restriction (or another joint constraint on assignments), indirect bidder complementarities cannot occur in assignment stages.

The bidders' desire for contiguous spectrum—or the auctioneer's restriction of winnings to be contiguous—is fundamentally inconsistent with *assignment substitutes*, a variation on the well-known gross substitutes property[5] which we prove is the "robust" condition for guaranteeing core outcomes in assignment stages. When the allocation and the assignment are determined together within a one-stage process, the unavoidable complementarities arising from contiguity may be completely overshadowed by the much larger values for generic blocks. As a result, the VCG outcome of the one-stage process may lie in the core, despite these complementarities. By way of contrast, the same unavoidable assignment complementarities, once isolated within an assignment stage under the two-stage approach, may stand out in stark relief. It then turns out that non-core VCG outcomes and all related anomalies are not merely theoretical possibilities, but rather are systematic attributes of the assignment stage.

Imposing the contiguity restriction dramatically reduces the number of feasible assignments. This reduction has two major implications. First, it becomes possible to explain any set of assignment bids with additive assignment values. Unfortunately, such well-behaved preferences by themselves do not imply core outcomes unless extra restrictions are imposed. Second, and independently from assignment preferences, the contiguity restriction eliminates many opportunities for bidder complementarities, and is shown to eliminate all of them in symmetric assignment stages (i.e., when each bidder entering the assignment stage has won the same number of blocks).[6] However, in asymmetric assignment stages with $n \geq 3$ bidders, non-core outcomes are generally unavoidable.

To provide a further comparison between assignment stages with and without the contiguity restriction, we introduce a concept of *detectable core violations*—a simple check that verifies whether a bidder is a "victim" of a non-core outcome (in the sense of being included in an unsatisfied blocking coalition) based exclusively on information about the auction outcome and the bidder's own bids. We show that, for unrestricted assignment stages, a bidder with submodular assignment values (e.g., one-block winners) will never encounter detectable core violations. In contrast, any bidder (including a one-block winner) with any non-degenerate assignment values can encounter detectable core violations as soon as a contiguity restriction is imposed.

[5] The gross substitutes condition was applied to tâtonnement processes by Arrow, Block, and Hurwicz (1959) and was introduced to auctions by Kelso and Crawford (1982).

[6] Under full symmetry, an assignment stage with the contiguity restriction is isomorphic to the (well-behaved) auction in which bidders have unit demands (Demange, Gale, and Sotomayor (1986)). For example, with six spectrum blocks and three two-block winners, the only feasible assignments give AB to one bidder, CD to another bidder and EF to the remaining bidder, yielding full substitutability and the corresponding core guarantees.

Finally, we provide a case study in all of the aforementioned issues by examining the assignment stage of one of the most conspicuous, recent spectrum auctions: the US Federal Communication Commission's Broadcast Incentive Auction of 2016–17. In particular, in this auction's assignment stage, three assignment areas exhibited zero-revenue outcomes and almost one-fifth of the assignment areas with at least three bidders yielded VCG outcomes that were outside the core. We propose an additional exercise for identifying assignment rounds that had the "potential" for non-core outcomes (in the sense that there existed bidder complementarities waiting to become relevant) and we find that it almost triples the incidence of anomalies (from 18% to 50%). Finally, we look for "winner collusion" and "loser collusion" opportunities for coalitions of bidders and we find them in 72% and 53% of assignment areas, respectively. As such, we provide the first empirical documentation of zero-revenue outcomes and related phenomena that arise from bidder complementarities in the VCG mechanism.

The remainder of this article is organized as follows. Section 2 reviews the motivation for two-stage approaches to spectrum auctions and the use of assignment stages. Section 3 provides examples of direct and indirect bidder complementarities. Section 4 introduces a stylized model of the two-stage approach for auctions. Section 5 considers assignment stages without the contiguity restriction. Section 6 develops a theory of assignment stages with enforced contiguity. Section 7 provides an empirical look at these issues in the assignment stage of the Broadcast Incentive Auction. Section 8 shows that other feasibility constraints can have similar effects as a contiguity requirement. Section 9 concludes.

2 The Motivation for Assignment Stages in Spectrum Auctions

There are two basic motivations for dichotomizing spectrum auctions into two stages, as practiced today. First, it simplifies bidding and speeds up the allocation process if bidders are enabled to indicate *quantities* of generic spectrum blocks instead of needing to bid for *specific frequencies*. This is particularly important in countries such as Australia, Canada and the US, where spectrum rights are partitioned into a large multiplicity of geographic licenses. Second, it improves efficiency if a mechanism is provided to enforce *contiguity* and thereby to eliminate a *holdout problem*.

The critical stylized fact behind the contiguity/holdout motivation is that, all other things being equal, bidders obtain enhanced value from contiguous frequencies. In spectrum auctions of the past decade, the simplest justification for this fact has been the LTE technology that is part of the 4G standard for mobile phones. Quite simply, a mobile provider obtains significantly greater data throughput from two adjacent blocks than from two noncontiguous blocks of the same size. However, the desire for contiguous spectrum goes well beyond LTE technology. The enhanced value of contiguity has been part of the spectrum valuation landscape since the 1990s—and mobile providers continue to desire contiguous spectrum for deployment of the newest 5G technology.

The desire for contiguous frequencies creates numerous possibilities for mischief by bidders in traditional one-stage spectrum auction designs. The simultaneous multiple round ascending (SMRA) auction, the prevalent format for the first 20 years of spectrum auctions, is especially conducive to anti-competitive strategies, due to bidding on specific frequencies. For example, in one real-world auction, "one bidder was attempting to assemble five adjoining licenses and the spoiler continued to bid on the middle license in a strategy that became known as 'giving the middle finger'."[7]

When licenses are close substitutes, splitting the auction into two stages provides an elegant solution to this holdout problem. Limited to bids for generic blocks during the allocation stage and limited to bids for contiguous assignments in the assignment stage, bidders are unable to disrupt winnings of others by strategically positioning themselves in the middle of attempted agglomerations.

These contiguity/holdout concerns and potential remedies are by no means exclusive to auctions to sell. The Broadcast Incentive Auction was made possible only after the FCC found an elegant solution to a corresponding holdout problem in a reverse auction.[8] The FCC wished to repurpose broadcast television spectrum to broadband data—but, for the spectrum to be useful, there needed to be a contiguous swath of spectrum cleared across the US. Since the spectrum was to be cleared on a voluntary basis, the FCC faced the likelihood of clearing only a checkerboard of spectrum. Avoiding a holdout problem while respecting the TV stations' property rights was accomplished by authorizing the FCC to "repack" the broadcast rights: each TV station maintained rights to generic frequency in its broadcast band, without rights to any specific frequency. In other words, to limit the ability of station owners to hold out and disrupt the contiguity of the spectrum it was buying, the FCC procured generic frequencies first, and assigned specific frequencies to the remaining stations later—a procedure that is strongly reminiscent of the two-stage approach.

There are additional practical disadvantages associated with traditional one-stage spectrum auction designs—with bidding on specific licenses—in settings with close substitutes. First, they frequently produce inordinately long auctions (with m licenses, it might take up to m rounds to increase the standing price on each license by just one price increment under the traditional SMRA format).[9] Conversely, two-stage approaches with generic licenses admit a single price for the m licenses; then, only a single price needs to be incremented, requiring only a single round and drastically shortening the duration. Second, one-stage designs run a higher risk of inefficiency due to the fitting problem. Since bidders need simultaneously to discover both the "right" quantity allocation (constituting a larger portion of the auction value) and the "right" assignment (constituting a smaller portion of the auction value), a failure to coordinate on the assignment ("the fitting problem") might compromise discovering the "right" quantity of winnings and lead to significant welfare losses. By

[7] McAfee, McMillan, and Wilkie (2010), p. 181.

[8] Federal Communications Commission (2010).

[9] As a result, the 2008 Canadian AWS auction extended 331 rounds and took 2 months. The 2014–15 US AWS-3 auction extended 441 rounds and took 2.5 months.

way of contrast, in the two-stage approach, the fitting problem never interferes with determining an optimal quantity allocation. Lastly, perhaps the strongest argument against the traditional approach is that the bidding language is not commensurate with the setting. When licenses are close substitutes for bidders, the need to select specific licenses becomes an artificial impediment that adds unnecessary complexity to bidding decisions.

At the same time, there are also challenges associated with the two-stage approach. The most obvious challenge is faced by bidders when licenses prove not to be close substitutes. If generic licenses encompass frequencies with significant value differences, bidders are exposed to considerable uncertainty when bidding for generic blocks, since they are essentially bidding (and paying) for items of unknown value in the allocation stage. In practice, auctioneers routinely engage in pre-auction product design. Licenses are organized into distinct categories of generic blocks when, for example, some of the frequencies are encumbered, in order to mitigate this problem and ensure that the "close substitutes" assumption is justified. A less obvious challenge is the proclivity of assignment stages to create bidder complementarities, which is at the heart of this article.

The first known instance of an assignment stage occurred in Trinidad and Tobago's GSM spectrum auction of June 2005. In the first, clock stage of the auction, up to two winners would be selected. In the second, assignment stage, the physical spectrum frequencies assigned to the winners would be determined.[10] Since then, assignment stages have become prevalent in spectrum auctions worldwide. As of this writing, four of the five most recent US spectrum auctions allocating specific frequencies were two-stage auctions that included assignment stages. Similarly, most recent Australian, Canadian and UK spectrum auctions have included assignment stages.

3 Bidder Complementarities in the Assignment Stage

This section uses simple examples to illustrate how bidder complementarities may arise in the assignment stage of a spectrum auction, in either of two ways. Consider a setting with four blocks (A, B, C and D) and three bidders each demanding at most two blocks, whose values are reported in Table 1.

First, consider allocating these blocks in a single stage using the VCG mechanism. The efficient assignment consists of awarding block A to Bidder 1, block B to Bidder 2 and blocks C and D to

[10] `https://tatt.org.tt/AboutTATT/SpectrumManagement/FirstSpectrumAuction2005.aspx` (accessed on December 16, 2022). The assignment stage of Trinidad and Tobago's auction utilized a first-price menu auction. Observe that, unlike VCG mechanisms, first-price menu auctions always generate core outcomes relative to the submitted bids. In the complete-information environments and with the refinement of coalition-proof Nash equilibrium considered by Bernheim and Whinston (1986), the outcomes of first-price menu auctions are efficient. However, this result clearly does not extend to incomplete-information environments—and, if a first-price menu auction is used to resolve the LLG model described in Section 1, local bidders 1 and 2 face a severe free-rider problem when jointly competing against the global bidder 3 (Ausubel and Baranov (2020)).

Table 1: *Example 1 — Direct Bidder Complementarity*

	Bidder 1	*Bidder 2*	*Bidder 3*
Values*	$\underline{v}_1(1) = 150$	$\underline{v}_2(1) = 100$	$\underline{v}_3(1) = 300$
	$v_1(A) = 170$	$v_2(B) = 120$	$\underline{v}_3(2) = 500$
	$\underline{v}_1(2) = 200$	$\underline{v}_2(2) = 200$	$v_3(AB) = 520$
VCG Outcome	$(A, 100)$	$(B, 50)$	$(CD, 110)$
Induced Assignment Stage			
Efficient Allocation in Stage 1	1	1	2
Assignment Values in Stage 2	$v_1(A\|1) = 20$	$v_2(B\|1) = 20$	$v_3(AB\|2) = 20$

* For Bidder 1, the value for the A block is 170, the value for any other block (B, C or D) is 150, and the value for any two blocks is 200. The values for Bidders 2 and 3 are interpreted analogously.

Bidder 3. It is easy to verify that the corresponding VCG payments are sufficiently high for the outcome to be in the core. Now consider awarding these blocks using the two-stage approach. To avoid unnecessary details, we do not specify the mechanism used for the allocation stage. Instead, we simply assume that the allocation stage ends with the efficient allocation of generic blocks: Bidders 1 and 2 get one block each, and Bidder 3 gets two blocks.

By the rules of the assignment stage, all bidders are guaranteed to receive assignments corresponding to their winnings from the allocation stage. Then the only assignment value that Bidder 1 has in the assignment stage is an incremental value of 20 for obtaining block A. This value corresponds to its true marginal value of obtaining block A instead of any other block. Similarly, the assignment values of the other bidders are Bidder 2's incremental value of 20 for block B and Bidder 3's incremental value of 20 for the AB combination. Given these values, it is evident that Bidders 1 and 2 are no longer competing against each other. Instead, they have now become complementary bidders who jointly compete against Bidder 3 in a classic LLG scenario, in which Bidders 1 and 2 act as local bidders and Bidder 3 acts as a global bidder. We refer to this scenario as a *direct* bidder complementarity since assigning either block A to Bidder 1 or block B to Bidder 2 physically conflicts with assigning AB to Bidder 3. With such a complementarity, if the auctioneer were to use the VCG mechanism for this assignment stage, Bidders 1 and 2 would have won their preferred assignments and paid zero—an obvious core violation given the positive bid of Bidder 3. Note that this example and the conclusion hold with or without the contiguity requirement on multi-block winnings, an important distinction emphasized in the next example.

Next, let us modify Example 1 with the values reported in Table 2. In particular, the only changes that are made in Example 2 (relative to Example 1) are an increase in value for the AB combination for Bidder 3 from 520 to 525, and a replacement of block B for block C in the value function of

Bidder 2. The new efficient assignment consists of awarding block D to Bidder 1, block C to Bidder 2 and blocks AB to Bidder 3. Again, it is easy to verify that the outcome of the one-stage VCG mechanism is squarely in the core.

Table 2: *Example 2 — Indirect Bidder Complementarity*

	Bidder 1	*Bidder 2*	*Bidder 3*
Values*	$\underline{v}_1(1) = 150$	$\underline{v}_2(1) = 100$	$\underline{v}_3(1) = 300$
	$v_1(A) = 170$	$v_2(C) = 120$	$\underline{v}_3(2) = 500$
	$\underline{v}_1(2) = 200$	$\underline{v}_2(2) = 200$	$v_3(AB) = 525$
VCG Outcome	$(D, 80)$	$(C, 50)$	$(AB, 130)$
	Induced Assignment Stage		
Efficient Allocation in Stage 1	1	1	2
Assignment Values in Stage 2	$v_1(A\|1) = 20$	$v_2(C\|1) = 20$	$v_3(AB\|2) = 25$

* For Bidder 1, the value for the A block is 170, the value for any other block (B, C or D) is 150, and the value for any two blocks is 200. The values for Bidders 2 and 3 are interpreted analogously.

Analogous to Example 1, we analyze the assignment stage of the two-stage approach after the allocation stage that induced the efficient allocation of generic blocks. First, suppose that there is no contiguity restriction on multi-block winnings. In this case, Bidder 2's demand for block C does not overlap with other demands, implying that no bidder complementarities are present. If the auctioneer were to use the VCG mechanism for this assignment stage, Bidder 3 would have won the AB assignment and paid 20, the opportunity cost imposed by Bidder 1, while Bidder 2 would have received block C for free—a fully competitive outcome.

Next, suppose that the auctioneer enforces the contiguity restriction that has been routinely imposed (for persuasive reasons, as recalled in Section 2) in recent spectrum auctions. Since Bidder 3 is required to win two adjacent blocks, it is now impossible for Bidders 1 and 2 to win their desired blocks simultaneously. As a result, Bidders 2 and 3 are now complementary bidders who jointly compete against Bidder 1. This example is unusual because it is driven by the contiguity restriction rather than by the global bidder's complementarity for goods. Indeed, in this recast LLG scenario, Bidder 1 (a one-unit winner) has effectively taken the role of "global" bidder and Bidder 3 (the two-unit winner) has effectively become a "local" bidder. We refer to this scenario as an *indirect* bidder complementarity, since assigning block C to Bidder 2 does not physically conflict with assigning block A to Bidder 1 (but it does so indirectly through the contiguity restriction). If the auctioneer were to use the VCG mechanism for this assignment stage, Bidders 2 and 3 would have won their preferred assignments and paid zero—an obvious core violation given the positive bid of Bidder 1.

Note that both examples above also illustrate how relatively small assignment complementarities can be "swamped" by much larger bidder values for generic blocks in a one-stage formulation, ensuring that there are no complementarities among bidders and that the outcome of the one-stage VCG mechanism is in the core. By contrast, the same small complementarities, direct or indirect, stand out in stark relief once the assignment is isolated from the allocation under the two-stage approach. Thus, dichotomizing a spectrum auction into two stages may focus attention on otherwise-irrelevant complementarities.

4 The Model

An auctioneer seeks to auction m indivisible blocks of spectrum denoted by set $M = \{1, ..., m\}$. The set of all possible combinations of blocks (i.e., packages) is denoted by $2^M = \{ z | z \subseteq M \}$. The auctioneer allocates these blocks to a set of bidders $N = \{1, ..., n\}$. For each bidder $i \in N$, the bidder's preferences over packages are characterized by a value function $v_i(\cdot)$. The value of the null package, $\emptyset$, is normalized to zero. The assumptions made about value functions are as follows:

(A1) Pure Private Values: Each bidder knows its own value for any package and this value is not affected by the values of other bidders or the packages allocated to other bidders;

(A2) Quasilinear Utility: Each bidder i's payoff from winning package z and making a payment of p is given by $v_i(z) - p$.

The restrictions on possible winnings imposed by the auctioneer (if any) are embedded into a set of admissible packages $\Omega \subseteq 2^M$. For example, the auctioneer can limit the number of blocks that can be won by individual bidders or restrict multi-block winnings to form a contiguous swath of spectrum when blocks are adjacent to each other.[11]

When bidders view blocks in M as being sufficiently similar to each other (in other words, when each bidder cares mostly about the quantity it wins, rather than the specific blocks), the auctioneer can simplify the auction process by dichotomizing it into two stages. In the first (allocation) stage, all blocks in M are treated as a homogeneous good and bidders bid on quantities of generic blocks. The winning allocation in terms of generic blocks and associated payments is determined using some auction mechanism. The central subject of this article is the second (assignment) stage of the auction, while the exact auction mechanism used for the allocation stage is out of scope.

In the assignment stage, each first-stage winner submits bids for all admissible packages in Ω consisting of the number of blocks it won in the first stage. Bidding in the assignment stage is optional since all winners of the allocation stage are necessarily assigned a quantity of specific blocks corresponding to their first-stage winnings. The winning assignments and associated assignment

[11] Note that the set of admissible packages Ω can be bidder-specific if needed.

payments are determined by the standard rules of the VCG mechanism, where the interpretation of counterfactual opportunity costs is not that bidder i is absent from the assignment stage, but merely that bidder i bids zero for all of its assignment options.

Formally, let Y denote the set of all possible quantity allocations of generic blocks among bidders in N, i.e.,

$$Y \;=\; \left\{ (y_1, ..., y_n) : \quad y_i \in \{0, ..., m\} \quad \forall i \in N \quad \text{and} \quad \sum_{j \in N} y_j \;\leq\; m \right\} . \tag{4.1}$$

Let $N(y) \;=\; \{\, i \in N : \; y_i > 0 \,\}$ and $n(y) = |N(y)|$ denote the set and the number of bidders who won at least one generic block in the allocation stage that led to assignment stage y, respectively. For the rest of the article, we refer to an assignment stage for assigning profile of generic winnings $y \in Y$ as an assignment stage y. For the ease of notation, we suppress indicator y where possible.

Let $\Omega(q)$ denote the set of all bundles that are consistent with quantity q, i.e.,

$$\Omega(q) \;=\; \left\{ (z_1, ..., z_m) \in \Omega : \; \sum_{k=1}^{m} z_k \;=\; q \right\} , \tag{4.2}$$

and let X denote the set of all feasible assignments for the assignment stage y, i.e.,

$$X \;=\; \{\, (x_1, ..., x_n) : \quad x_i \in \Omega(y_i) \quad \text{and} \quad x_i \cap x_j = \emptyset \quad \forall i, j \in N : i \neq j \,\} . \tag{4.3}$$

For each bidder i, let $\underline{v}_i(q)$ and $\bar{v}_i(q)$ denote the value of bidder i's least preferred and the most preferred assignment options corresponding to q generic blocks, i.e.,

$$\underline{v}_i(q) \;=\; \min_{z \in \Omega(q)} v_i(z) \qquad \text{and} \qquad \bar{v}_i(q) \;=\; \max_{z \in \Omega(q)} v_i(z) , \tag{4.4}$$

and let $\delta_i(\cdot|q)$ denote bidder i's marginal value for assignments of q blocks relative to its least preferred option, i.e.,

$$\delta_i(z|q) \;=\; v_i(z) \;-\; \underline{v}_i(q) \qquad \forall z \in \Omega(q) . \tag{4.5}$$

For any bidder coalition $C \subseteq N$, the coalitional value function on a subset of feasible assignments $X' \subseteq X$ is given by:

$$w_{C}(X') \;=\; \max_{x \in X'} \sum_{j \in C} \delta_j(x_j|y_j) . \tag{4.6}$$

Note that the above definition does not treat bidders outside coalition C as being absent, but merely ignores their assignment values while still assigning them to feasible assignment options.

For an assignment stage y, assignment $x^e \;=\; (x_1^e, ..., x_n^e) \in X$ is efficient if

$$\sum_{j \in N} \delta_j(x_j^e|y_j) \;=\; w_{N}(X) . \tag{4.7}$$

Obviously, obtaining efficiency in the assignment stage does not imply the standard notion of allocative efficiency relative to value functions $\{v_i(\cdot)\}_{i=1}^n$ unless quantity profile y consists of efficient quantities of generic blocks.

Let X^e_C denote the set of all feasible assignments in which bidders in coalition C are assigned their efficient assignments, i.e.,

$$X^e_C \;=\; \left\{ x \in X : \; x_j = x^e_j \quad \forall j \in C \right\} \; . \tag{4.8}$$

Observe that due to definitions (4.6)–(4.8),

$$w_{C}(X^e_C) \;=\; w_{C}(X^e_{N\backslash C}) \qquad \forall C \subseteq N \; . \tag{4.9}$$

For assignment stage y, a Vickrey outcome consists of an efficient assignment x^e and a payment vector $p^V \;=\; (p^V_1, ..., p^V_n)$ such that

$$p^V_i \;=\; w_{N\backslash i}(X) \;-\; w_{N\backslash i}(X^e_{N\backslash i}) \qquad \forall i \in N \; , \tag{4.10}$$

and a core outcome consists of an efficient assignment x^e and a payment vector $p \;=\; (p_1, ..., p_n)$ belonging to the set of core payments CP defined as:

$$CP \;=\; \left\{ p \in \mathbb{R}^n_+ : \;\; w_{C}(X) \;-\; w_{C}(X^e_C) \;\leq\; \sum_{N\backslash C} p_j \;\leq\; w_{N\backslash C}(X^e_{N\backslash C}) \quad \forall C \subset N \right\} . \tag{4.11}$$

Both the Vickrey outcome and core outcomes with respect to bids are defined in the analogous way by substituting bids for values.

In this article, we do not model the first (allocation) stage of the two stage approach. Instead, we simply assume that the winning allocation of generic blocks is given by $y^* \in Y$. The second (assignment) stage is modeled as a Vickrey auction. Each bidder i submits bid $b_i(z)$ for each assignment option $z \in \Omega(y^*_i)$. The winning assignment x^* is a solution to:

$$x^* \in X^* \;=\; \arg\max \sum_{j \in N} b_j(x_j) \qquad \text{s.t.} \quad x \in X(y^*) \; . \tag{4.12}$$

If X^* contains multiple solutions, the auctioneer selects x^* according to some pre-determined tie-breaking procedure. The assignment payment p^*_i for each bidder i is calculated using Vickrey formula (4.10) by substituting marginal values $\delta_j(x_j|y_j)$ with the corresponding bids $b_j(z)$ for each assignment option $z \in \Omega(y^*_j)$ and each bidder $j \in N$. The assignment payoff of bidder i (fully attributed to the assignment stage) is given by $\pi^*_i \;=\; \delta_i(x^*_i|y^*_i) - p^*_i$. Ultimately, after two stages, bidder i wins bundle x^*_i and pays the sum of two payments, one payment from the first stage for winning y^*_i generic blocks and assignment payment p^*_i from the second stage for obtaining x^*_i. Note that all definitions above include all bidders in set N, even bidders who did not win any generic blocks in the allocation stage and, consequently, do not actively participate in the assignment stage. This is justified since the above definitions imply that $x_i^* = \emptyset$ and $p_i^* = 0$ for any bidder with $y_i^* = 0$.

The Vickrey payment rule in the assignment stage incentivizes bidders to bid truthfully on all feasible assignment options. For assignment stage y, truthful bidding for bidder i consists of bidding zero for its least preferred assignment option (since the bidder is guaranteed an assignment) and bidding its true marginal value for any other available options, i.e.,

$$b_i(z) \;=\; \delta_i(z|q) \;=\; v_i(z) \;-\; \underline{v}_i(q) \qquad \forall z \in \Omega(y_i) \;. \tag{4.13}$$

Due to well-known properties of the Vickrey auction, the following statements are true for any assignment stage y:

1. Truthful bidding is a weakly dominant strategy for all bidders, implying the efficiency of winning assignment x^*; and
2. When $n(y) \leq 2$, the Vickrey outcome of the assignment stage is always in the core with respect to values and bids.

Examples 1 and 2 from Section 3 have already established our first finding:

Proposition 1. *A set of value functions satisfying assumptions (A1) and (A2) and such that the VCG outcome of the (one-stage) VCG mechanism is in the core may nonetheless induce, under the two-stage approach, an assignment stage in which the VCG outcome lies outside the core with respect to assignment values.*

Our main theoretical contributions are presented next. In Section 5, we study properties of assignment stages without the contiguity restriction. Then, in Section 6, we consider assignment stages with the contiguity restriction, a standard way in which they are implemented in practice.

5 Unconstrained Assignment Stages

Throughout this section, we assume that the auctioneer does not impose the contiguity (or any other) restriction and, so, the set of admissible packages is unconstrained (i.e., $\Omega \;=\; 2^M$).

The core guarantees of the VCG mechanism are closely related to two well-known notions of diminishing returns for bidders' preferences: submodularity and gross substitutes. The value function $v(\cdot)$ is *submodular* if for any two bundles $z, z' \subseteq M$ such that $z \subseteq z'$ and for every item $k \in M \setminus z'$,

$$v(z+k) \;-\; v(z) \;\geq\; v(z'+k) \;-\; v(z') \;. \tag{5.1}$$

The gross substitutes property is stated in terms of the bidder's demand correspondence,

$$x(p) \;=\; (x_1(p), ..., x_m(p)) \;=\; \arg\max_{z \in \Omega}\{\, v(z) \;-\; p \cdot z \,\} \;, \tag{5.2}$$

and the set of nonnegative price vectors P at which $x(\cdot)$ is single-valued. The value function $v(\cdot)$ satisfies *gross substitutes (GS)* if, for any two price vectors $p, p' \in P$ such that $p \leq p'$ and every $k \in M$ for which $p_k = p'_k$,

$$x_k(p') \geq x_k(p) . \tag{5.3}$$

It is well-known that gross substitutes implies submodularity (Gul and Stacchetti (1999)), and that bidders' preferences satisfying gross substitutes is sufficient and "almost necessary" for the outcome of the VCG mechanism to be in the core (Ausubel and Milgrom (2002)).

In the assignment stage, bidders' bids are restricted to bundles with a constant number of blocks (corresponding to generic winnings from the allocation stage), so we have to properly adjust the notions of submodularity and gross substitutes.

Let $\Omega(q+)$ denote the set of all admissible bundles with at least q blocks, i.e.,

$$\Omega(q+) = \left\{ (z_1, ..., z_m) \in \Omega : \sum_{k=1}^{m} z_k \geq q \right\} . \tag{5.4}$$

A bidder's *assignment value function* for quantity q is defined as follows. For each bundle $z \in \Omega(q+)$, $v(z|q)$ is given by the highest incremental value that the bidder can obtain from selecting exactly q blocks out of blocks contained in bundle z relative to $\underline{v}(q)$, i.e.,

$$v(z|q) = \max_{z' \in \Omega(q)} \{v(z') : z' \leq z\} - \underline{v}(q) \qquad \forall z \in \Omega(q+) . \tag{5.5}$$

Observe that only values for bundles of size q are used to construct the assignment value function $v(\cdot|q)$, and that $v(z|q) = \delta(z|q)$ for all $z \in \Omega(q)$. The bidder's demand correspondence $x(p|q)$ based on $v(\cdot|q)$ is defined analogously to (5.2).

The corresponding notions of submodularity and gross substitutes for assignment stages are defined as follows.

Definition 1. *The value function $v(\cdot)$ is said to be assignment submodular (satisfy assignment substitutes) for quantity $q \in \{0, ..., m\}$ if the assignment value function $v(\cdot|q)$ is submodular (satisfies gross substitutes) on domain $\Omega(q+)$.*

The notions of submodularity and assignment submodularity are different and neither one implies the other: submodularity (5.1) puts restrictions on values of bundles of different sizes, while assignment submodularity puts restrictions on values of bundles of the same size. Assignment substitutes for quantity q implies assignment submodularity for the same quantity.[12] At the same time, assignment substitutes for all quantities (which restricts values on bundles of the same size) obviously

[12] This follows from Gul and Stacchetti (1999). An alternative proof is included as part of the proof for Proposition 2 in the appendix.

cannot imply gross substitutes (which applies to all bundles).[13] Finally, any function $v(\cdot)$ satisfies assignment substitutes for $q = 0$ (trivially) and for $q = 1$ since $v(\cdot|1)$ is a unit-demand valuation. Our next result shows that gross substitutes implies assignment substitutes for all quantities.

Proposition 2. *If the value function $v(\cdot)$ satisfies gross substitutes, then it satisfies assignment substitutes for all quantities $q = 0, ..., m$.*

Recall that for all $i \in N$ and any $z \in \Omega(q)$, $v_i(z|q) = \delta_i(z|q)$. It immediately follows that the definition of coalition value function (4.6) can be restated using the assignment value functions $\{v_i(\cdot|\cdot)\}_{i=1}^n$ as follows:

$$w_C(X') = \max_{x \in X'} \sum_{j \in C} v_j(x_j|y_j) \qquad \forall X' \subseteq X \qquad \forall C \subseteq N \; . \tag{5.6}$$

A standard definition of the coalitional value function is stated in terms of blocks available to the coalition rather than the set of feasible assignments. Let $X_C(z)$ denote the set of all feasible assignments in which bidders in coalition $C \subseteq N$ are limited to blocks contained within bundle $z \subseteq M$, i.e.,

$$X_C(z) = \left\{ x \in X : \sum_{j \in C} x_j \leq z \right\} . \tag{5.7}$$

Then, the coalitional value function of coalition $C \subseteq N$ for bundle $z \subseteq M$ is given by

$$\widetilde{w}_C(z) = \begin{cases} w_C(X_C(z)) & X_C(z) \neq \emptyset \\ -\infty & X_C(z) = \emptyset \end{cases} , \tag{5.8}$$

and its effective domain is given by

$$\Omega_C = \{ z \in 2^M : X_C(z) \neq \emptyset \} . \tag{5.9}$$

The corresponding notions of assignment submodularity and assignment substitutes for coalitional value functions are defined symmetrically.

Definition 2. *For assignment stage $y \in Y$ and coalition C, the coalitional value function $\widetilde{w}_C(\cdot)$ is assignment submodular (satisfies assignment substitutes) if it is submodular (satisfies gross substitutes) on the Ω_C domain.*

For the standard setting of a package auction, bidders' values satisfying gross substitutes implies gross substitutes for their coalitional value functions (Ausubel and Milgrom (2006), Theorem 8). The corresponding result also holds in the assignment setting.

[13] Consider an example with a bidder who (1) does not have any preferences for specific block assignments, but (2) has marginal values for generic blocks that violate decreasing marginal returns. While assignment substitutes are satisfied for all quantities, gross substitutes is obviously violated.

Proposition 3. *For an assignment stage $y \in Y$ and coalition $C \subseteq N$, if $v_i(\cdot)$ satisfies assignment substitutes for quantity y_i for all $i \in C$, the coalitional value function $\widetilde{w}_C(\cdot)$ satisfies assignment substitutes.*

Naturally, when $\widetilde{w}_C(\cdot)$ satisfies assignment substitutes, it must be assignment submodular. Our next result shows that Vickrey payments must satisfy the core constraint for any coalition with a submodular coalitional value function.

Proposition 4. *Consider an assignment stage $y \in Y$ and coalition $C \subseteq N$ with coalitional value function $\widetilde{w}_C(\cdot)$ that is assignment submodular. Then the Vickrey payment vector p^V satisfies the core constraint for coalition C, i.e.,*

$$w_C(X) - w_C(X^e_C) \leq \sum_{N \backslash C} p^V_j \leq w_{N \backslash C}(X^e_{N \backslash C}) . \tag{5.10}$$

Propositions 3 and 4 together imply:

Corollary 1. *For an assignment stage $y \in Y$, the Vickrey outcome is in the core with respect to the bidders' true assignment values when*

(a) $n(y) = 3$ and the value function $v_i(\cdot)$ of each bidder $i \in N(y)$ is assignment submodular for quantity y_i; or

(b) $n(y) \geq 4$ and the value function $v_i(\cdot)$ of each bidder $i \in N(y)$ satisfies assignment substitutes for quantity y_i.

Proof. Part (b) is straightforward. For part (a), note that the only core constraints that can be violated by the Vickrey payments are for coalitions of size 1 ($|C| = 1$). As a result, assignment substitutes is unnecessary and assignment submodularity is fully sufficient. □

Remark. For the standard setting of a package auction, versions of Proposition 4 and Corollary 1 can be proved analogously. As such, they provide an alternative (and more direct) proof of the well-known result that bidders' preferences satisfying gross substitutes necessarily place the Vickrey outcome in the core (Ausubel and Milgrom (2002)).

Following Corollary 1, a general sufficient condition for the Vickrey outcome of every possible assignment stage to be in the core is all value functions $\{v_i(\cdot)^n_{i=1}\}$ satisfying assignment substitutes for each quantity $q \in \{0, ..., m\}$. Then, by Proposition 2, all value functions $\{v_i(\cdot)^n_{i=1}\}$ satisfying gross substitutes is one possible sufficient condition.

The literature on the VCG mechanism and core-selecting auctions has traditionally emphasized the negative effect of non-core outcomes on the auctioneer (e.g., low revenue). However, non-core outcomes can also be harmful to unsatisfied bidders who form blocking coalitions with the auctioneer—such bidders are victims as well in the sense that their opponents get away with valuable items (that they wanted) without paying. This might be particularly concerning in spectrum auctions where bidders compete with each other in the downstream cellular service markets, and an unsatisfied entrant may find itself in a weakened long-term market position.

Is it possible for a bidder to ensure that its own opportunity costs are fully reflected in opponents' payments (in the sense of never appearing on the blocking side of a violated core constraint)? To provide some answers, we introduce a new concept of *detectable* core violations.

Definition 3. *Bidder i is said to encounter a detectable core violation if there exists coalition $C \subseteq N$ such that bidder $i \in C$ and*

$$\sum_{N \setminus C} p_j^V \; < \; w_i(X^e_{C\setminus i}) \; - \; w_i(X^e_i) \; . \tag{5.11}$$

A core constraint (5.10) tests whether bidders in coalition $N\setminus C$ pay enough to cover the opportunity costs of coalition C, while inequality (5.11) tests whether bidders in $N \setminus C$ pay enough to cover the opportunity costs of just bidder i (assuming that bidders in $C \setminus i$ impose zero opportunity costs on bidders in $N \setminus C$). By construction,

$$\begin{aligned} w_C(X) \; - \; w_C(X^e_C) \; & \geq \; w_{C\setminus i}(X^e_{C\setminus i}) \; + \; w_i(X^e_{C\setminus i}) \; - \; w_{C\setminus i}(X^e_{C\setminus i}) \; - \; w_i(X^e_i) \\ & = \; w_i(X^e_{C\setminus i}) \; - \; w_i(X^e_i) \; , \end{aligned}$$

so a detectable core violation for bidder i and coalition C immediately implies an actual core violation for coalition C.

We refer to core violations (5.11) as detectable in the following sense. Suppose that the auctioneer runs a non-transparent auction in which only the auction outcome (winnings and payments) is disclosed to bidders. In order to check for core violations (5.10), a bidder needs to know the other bidders' actual bids (which are unavailable). In contrast, checking (5.11) only requires knowledge of the auction outcome and the bidder's own bids. When present, these core violations are visible to a bidder and, thus, detectable. In the next proposition, we show that a bidder with well-behaved assignment values never encounters detectable core violations.

Proposition 5. *Consider an assignment stage $y \in Y$ and bidder $i \in N$ with value function $v_i(\cdot)$ that is assignment submodular for quantity y_i. Then bidder i never encounters any detectable core violations, i.e., for any coalition $C \subseteq N$ such that $i \in C$*

$$\sum_{N \setminus C} p_j^V \; \geq \; w_i(X^e_{C\setminus i}) \; - \; w_i(X^e_i) \; . \tag{5.12}$$

To summarize, submodularity (or substitutability) of a bidder's own values cannot guarantee protection against core violations (of which the bidder is a victim), as such guarantees depend on the values of all bidders (Propositions 3 and 4). However, it does guarantee protection against detectable core violations for the bidder (Proposition 5).

It is worth pointing out that any bidder who has won a single block in the allocation stage is automatically protected against violations of (5.12) since any assignment values for $q = 1$ satisfy submodularity. In the next section, we consider assignment stages with the contiguity restriction, the standard way in which they are implemented in practice. Surprisingly, we will show that enforcing contiguity eliminates all guarantees of (5.12), even for winners of single blocks!

6 Adjacent Blocks and Contiguous Assignments

In this section, we consider assignment stages in which blocks in M are physically adjacent to each other as in a typical band plan of a spectrum auction, and in which the auctioneer restricts any multi-block winnings to be contiguous assignments. Formally, we assume that all blocks are taken from a contiguous band of spectrum such that block 1 is the bottom end of the band, block m is the top end of the band, and any interior block $j = 2, ..., m-1$ is adjacent to blocks $j-1$ and $j+1$. All bidders in N are constrained to win contiguous assignments (i.e., bundles in which the assigned blocks form a connected set), which is captured by the following description of the set of feasible bundles:

$$\Omega \;=\; \left\{ (z_1, ..., z_m) \in 2^M : \quad z_k z_{k'} = 1 \;\text{ for } k < k' \implies \quad z_s = 1 \quad \forall s \in \{k, ..., k'\} \right\} \tag{6.1}$$

Given our earlier definitions, it follows that any bidder i who has won $q \geq 2$ blocks in the allocation stage is guaranteed/restricted to win an assignment that consists of q adjacent blocks (a contiguous assignment). For ease of exposition, we limit attention in this section to assignment stages in which all generic blocks have been allocated to bidders in N (i.e., there are no unsold blocks).

We start our analysis with a simple observation that the contiguity restriction and assignment substitutes cannot coexist. To see this, consider a bidder who has won two generic blocks in the allocation stage (out of blocks A, B, C and D) and has an extra marginal value of 20 for any assignment that includes block A. Without the contiguity restriction, such values satisfy assignment substitutes. But, with the contiguity restriction, such bidder has value of 20 for the AB combination, and zero for the BC and CD assignments. If the bidder demands AB at an initial price vector, its demand will switch to CD when the price of B is made sufficiently high, causing demand to decline for block A—a clear violation of assignment substitutes. Moreover, observe that the contiguity restriction has turned bidder preferences from exhibiting assignment substitutes to ones exhibiting a perfect complementarity between blocks A and B, inducing the exact bid pattern used in the examples of Section 3.

Our next observation is that, under the contiguity restriction, the number of feasible assignments for a bidder becomes quite small. Consider a bidder who won $q \geq 2$ generic blocks in the allocation stage. Without the contiguity restriction, the bidder has $\frac{m!}{q!(m-q)!}$ possible assignments. However, with the contiguity restriction, the same bidder has at most $m-q+1$ feasible assignment options, depending on the winnings of other bidders. The starkest implication of the contiguity restriction and the corresponding reduction in feasible options is the ability to express any set of assignment values using simple value functions. Observe that, with m available blocks and only $m-q+1$ assignment values, there are always enough degrees of freedom to explain any assignment values with an additive value function in which a bidder attaches a value (positive or negative) to each block and the value of any assignment is given by the sum of values of the blocks in the assignment.[14] In other words, we can limit attention to additive assignment preferences without any loss of generality.

With the contiguity restriction, additive assignment preferences cannot by themselves provide core guarantees, and we must consider further limitations on preferences. One possible restriction on additive valuations is to limit the number of non-zero block values to just one. When a bidder has a positive value for a single block (the "favored" block), the bidder values any contiguous assignment with the favored block at a constant premium to any contiguous assignment without the favored block. Alternatively, when a bidder has a negative value for a single block (the "poisoned" block), the bidder values any contiguous assignment without the poisoned block at a constant premium to any contiguous assignment with the poisoned block.[15]

The next proposition establishes that a single non-zero block value restriction does provide core guarantees, but only when the same block is either simultaneously favored by all bidders or simultaneously considered poisoned by all bidders.

Proposition 6. *Suppose that there are no unsold blocks, all bidders have additive assignment values with a single non-zero component and the contiguity restriction is enforced. If the same block is either simultaneously favored by all bidders or simultaneously viewed as poisoned by all bidders, then the VCG outcome of the assignment stage is in the core with respect to bidders' assignment values.*

The sufficient condition in Proposition 6 might be empirically relevant. Frequently, in assignment stages, a particular block in a band is encumbered by a prior user or receives interference from an

[14] Consider a bidder who won two generic blocks (out of blocks A, B, C and D). In the assignment stage with the contiguity restriction, the bidder can specify at most three assignment values, for AB, BC, and CD combinations. Suppose that these assignment values are given by $\delta(AB|2) = 0$, $\delta(BC|2) = 4$, $\delta(CD|2) = 2$. Then, one possible set of block values are $\{0, 0, 4, -2\}$ for A, B, C and D, correspondingly. These values can be used to recover a value function (up to a constant) for all bundles with two blocks.

[15] For example, consider a bidder who won two generic blocks (out of A, B, C and D) and favors block B by 10. The corresponding assignment values are 10 for AB and BC, and 0 for CD. Meanwhile, if block B is instead poisoned by 10, the corresponding assignment values are 10 for CD, and 0 for AB and BC.

adjacent spectrum band affecting all bidders in the same way (a poisoned block). Nevertheless, this condition is very special. For example, even when each bidder favors a single block but the favored block varies across bidders, non-core VCG outcomes become possible, as evidenced by the examples of Section 3 (both examples belong to this subclass).

Since the class of values in Proposition 6 is overly restrictive, we should also consider restrictions on the other primitive of the assignment stage, the allocation of generic blocks among bidders entering the assignment stage. We have already established that enforcing contiguity dramatically reduces the number of possible assignments in the band. It is easy to see that this reduction can potentially eliminate some opportunities for bidder complementarities. For example, consider a two-block winner who competes for the BC assignment (out of AB, BC and CD) and faces two one-block winners. Under the contiguity restriction, the bidder will never compete against both of them simultaneously, since it is not possible for one-block winners to win B and C jointly, and force the two-block winner to AD. Thus, independent of bidders' assignment preferences, core guarantees might arise naturally in settings in which the contiguity restriction eliminates bidder complementarities by itself. Our next proposition identifies such settings. We refer to assignment stage y in which all bidders in $N(y)$ have won the same number of generic blocks as *symmetric*, and otherwise *asymmetric*.

Proposition 7. *When there are no unsold blocks and the contiguity restriction is enforced, the VCG outcome is in the core with respect to bidders' bids in any symmetric assignment stage.*

Proof. Suppose that each bidder in $N(y)$ has won q blocks. With the contiguity restriction, the set of feasible assignment options for each bidder is fully captured by the bidder's *position* in the band (the first position corresponds to blocks $1, ..., q$, the second position corresponds to blocks $q + 1, ..., 2q$, etc.). With the same set of bidding options, the symmetric assignment problem is equivalent to assigning positions: each bidder has to be assigned one position from the set $\{1, 2, ..., n(y)\}$. In this formulation, all bidders have unit demands, trivially satisfying assignment substitutes and, by Corollary 1, ensuring that the VCG outcome is in the core. □

Conversely, for any asymmetric assignment stage with the contiguity restriction, it is trivial to generate scenarios with core violations.

Proposition 8. *Suppose that there are no unsold blocks and the contiguity restriction is enforced. For any asymmetric assignment stage y with $n(y) \geq 3$, it is possible to specify additive assignment values such that the VCG outcome is not in the core.*

Our findings so far indicate that assignment stages with the contiguity restriction, outside of two special conditions identified in Propositions 6 and 7, are prone to bidder complementarities and non-core VCG outcomes. Ironically, the desirability of contiguous assignments, one of the main motivations for the two-stage approach and for assignment stages in spectrum auctions, turns out to be fundamentally incompatible with core guarantees in the assignment stage.

Lastly, we turn to the question of whether a single bidder can ensure that its opponents always pay the full amount of its opportunity costs in the sense of not having detectable core violations. To state our result, we first need the following definition of strict assignment preferences.

Definition 4. *For assignment stage y, bidder i is said to have strict assignment preferences if all assignment options in $\Omega(y_i)$ can be strictly ranked from the most preferred assignment option z_1 to the least preferred assignment option z_l where $l = |\Omega(y_i)|$, i.e.,*

$$\bar{v}_i(y_i) = v_i(z_1) > v_i(z_2) > ... > v_i(z_l) = \underline{v}_i(y_i) .$$

In symmetric assignment stages, bidders never worry about being a victim of a non-core outcome since all core constraints are always satisfied due to contiguity. For asymmetric assignment stages, we have the following negative result.

Proposition 9. *Suppose that there are no unsold blocks and the contiguity restriction is enforced. For any asymmetric assignment stage y with $n(y) \geq 3$ such that bidder $i \in N(y)$ has strict assignment preferences, it is possible to specify assignment values for bidders in $N(y) \setminus i$ such that bidder i encounters a detectable core violation, with one exception: if $n(y) = 3$ with $y = (y_i, y_j, y_k)$ such that $y_j = y_k$ and the most preferred assignment for bidder i corresponds to the "middle" position in the band (the unique feasible assignment for bidder i that does not include end blocks 1 or m).*

The intuition behind the proof of Proposition 9 is instructive. The contiguity restriction simultaneously enables indirect bidder complementarities and prevents multi-block winners from expressing submodular assignment values, creating opportunities for direct bidder complementarities. Facing a possibility of both direct and indirect bidder complementarities, a bidder is never protected against violations of (5.12), even when bidding for a single block (with an exception of one "almost" symmetric setting).

Remark. One could have generalized Definition 3 to speak of when a *set* of bidders encounters a detectable core violation (e.g., if they are allowed to share their information with each other). In that event, there are immediate counterparts to Propositions 5 and 9 for sets of bidders.

Overall, we see that enforcing contiguity has an ambiguous effect on the likelihood of non-core outcomes. On the one hand, the contiguity restriction removes any chance of substitutes preferences for multi-block winners and enables indirect bidder complementarities. On the other hand, the dramatic reduction in feasible assignments eliminates some possibilities for bidder complementarities, including their total elimination in symmetric assignment stages.

In the next section, we will provide an empirical case study of bidder complementarities, non-core outcomes and related issues in the assignment stage of the FCC's Broadcast Incentive Auction of 2016-17.

7 The Assignment Stage of the Broadcast Incentive Auction

The FCC's Broadcast Incentive Auction included two separate but interdependent auctions: a reverse auction, where television broadcasters bid to relinquish their spectrum rights; and a forward auction, where wireless operators bid to acquire licenses for the freed-up spectrum.[16] For the forward auction, the FCC adopted a two-stage approach. In the first stage, bidders bid in a uniform-price ascending clock auction for generic blocks of spectrum in each of 416 distinct partial economic areas (PEAs). In the second stage, bidders bid for physical frequency assignments of their generic winnings.

The assignment stage was conducted as a sequence of sealed-bid auction rounds, organized in up to six parallel sessions. Bidders bid for their assignments independently in each region, in descending order of population. To accelerate the process and to enhance geographic contiguity, PEAs with the same winners and same winnings were grouped together into assignment areas.[17] As a result, the total number of assignment rounds was reduced from 416 to 228. Participating bidders were informed about their own assignments and payments in each round before they bid for assignments in the next round.

For the assignment stage, the FCC adopted the VCG mechanism with a contiguity requirement: each winner of generic blocks was guaranteed a contiguous assignment of spectrum within the assignment area. Bidders were invited to bid on all possible contiguous assignments corresponding to their generic winnings—even though some of these assignments would be incompatible with maintaining contiguity for other winners—in order to avoid disclosing information about the winnings of other bidders.

Before presenting the main results of our empirical analysis, we provide relative block values expressed in the submitted bids of the assignment stage. To limit interpretational issues, we initially look only at bids for single-block assignment options, i.e., bids by winners of one generic block; note that 2,884 bids (60% of the 4,753 assignment stage bids) were single-block bids.[18] Table 3 reports

[16] A detailed description of the Broadcast Incentive Auction can be found in Aperjis, Ausubel, and Baranov (2017). All publicly-available bidding data can be found at `https://auctiondata.fcc.gov/public/projects/1000`.

[17] When two or more PEAs had the same winners and the same generic winnings, they were combined into a single assignment area. In the assignment stage, a bidder would then bid for (and win) the same assignment option in all PEAs included in the assignment area.

[18] The restriction to single-block bids also has the effect of removing T-Mobile's bids from Table 3. T-Mobile won three or four blocks in most regions and bid so as to concentrate its holdings in the B, C and D blocks, so analyzing T Mobile's bids would have little effect on the conclusions drawn in this paragraph.

the average amount bid for each single-block assignment option, normalized by the product of the number of MHz and the population of the assignment area—the standard price measure used in spectrum valuation. The average bid amounts ranged from 0.11 cents (per MHz-pop) for the *A* block to 3.94 cents for the *F* block. By comparison, the average price paid for generic spectrum in the allocation stage was about $1 per MHz-pop, approximately 30x the assignment values. Interestingly, while average bids for blocks *B*–*G* were in a narrow range of 2.68 to 3.94 cents (per MHz-pop), the average bid for the *A* block was only 0.11 cents, indicating that bidders generally viewed the *A* block as the least desirable. One likely reason for this was that the *A* block lay adjacent to the guard band for TV channel 37, which is reserved for radio astronomy. As such, bidders knew that there would not be mobile spectrum available immediately adjacent to and below the *A* blocks—and they may have also had concerns that power restrictions would ultimately be placed on *A* blocks near radio observatories.

Table 3: *Average Bid Amounts for Single-Block Assignment Options*

	A	*B*	*C*	*D*	*E*	*F*	*G*
*Average Bid (cents/MHz*pop)*	**0.11**	3.73	2.91	3.11	3.71	3.94	2.68

The bidding data from the assignment stage of the Broadcast Incentive Auction presents us with a unique empirical opportunity to evaluate the extent of non-core outcomes in the VCG mechanism. We conduct five empirical exercises to identify actual and potential non-core outcomes, to find opportunities for winner and loser collusion, and to classify the nature of underlying bidder complementarities (direct or indirect).

Our first exercise is very straightforward and yet the results are rather remarkable. In the assignment stage of the Broadcast Incentive Auction, there were 18 assignment areas (out of 228) that generated zero revenues. In 15 of these instances, all bidders were able to obtain their most-preferred feasible assignments, confirming that these zero-revenue outcomes were fully competitive. But most notably, in each of the remaining three instances of zero revenues, there was a single bidder (the same one in all three instances) who failed to win its most preferred assignment, only to see its opponents (also the same ones in all three instances)[19] get their preferred assignments for free, a very detectable core violation. To the best of our knowledge, this is the first time that (non-core) zero-revenue outcomes of the VCG mechanism have been documented in the field.

We now provide details for one of these instances. The assignment of seven blocks in PEAs 224 and 287 (DeKalb, IL and Kenosha, WI, which had been grouped together) was made in assignment

[19] It is especially striking that each of these three instances involved exactly the same bidders, given that the identities of opponents in the assignment stage were kept anonymous until after the auction concluded.

round 37 (REAG 3). In this assignment area, Dish Network won one generic block, while T-Mobile and U.S. Cellular each won three generic blocks. All bids, including discarded incompatible bids, submitted by bidders in this round are reported in Table 4, with winning bids displayed in bold.

Table 4: *Assignment Bids for PEAs 224 and 287 in $*

Bandplan

A	*B*	*C*	*D*	*E*	*F*	*G*

Bidders (generic winnings)	Dish Network (1 block)	T-Mobile (3 blocks)	U.S. Cellular (3 blocks)
*Compatible Bids**	***A*, 0**	*ABC*, 0.4m	*ABC*, 0
(block(s), bid)	*D*, 0	***BCD*, 3.5m**	*BCD*, 0
	G, 0.237m	*DEF*, 0.01m	*DEF*, 4m
		EFG, 0	***EFG*, 8.7m**
*Incompatible Bids**	*B*, 0.308m	*CDE*, 2.1m	*CDE*, 0.01m
(block(s), bid)	*C*, 0		
	E, 0		
	F, 0.31m		
VCG payments	**0**	**0**	**0**

* A *compatible bid* is a bid for an assignment option that is compatible with the joint contiguity restriction. An *incompatible bid* is a bid for one that is not.

The mechanics of the VCG payment calculations for this example are as follows. U.S. Cellular's bid of $8.7 million for blocks EFG prevented Dish Network from winning block G. Independently, T-Mobile's bid of $3.5 million for blocks BCD also prevented Dish Network from winning block G, since winning it would have forced U.S. Cellular to a noncontiguous assignment. Therefore, taking away the bids of either T-Mobile or U.S. Cellular (but not both), Dish Network would still have been assigned block A. As a result, neither T-Mobile nor U.S. Cellular "caused" the assignment and so the VCG payments of both these bidders equaled zero, a non-core outcome resulting from an indirect bidder complementarity between the two bidders. Their joint payment would have needed to increase by $237,238 in order to be in the core. Furthermore, the outcome of this assignment round would have stayed the same even if Dish Network had bid substantially higher. In fact, the VCG payments for both T-Mobile and U.S. Cellular would have remained zero if Dish Network had bid as much as $3.1 million for block G.

In the two other assignment areas that generated uncompetitive zero-revenue outcomes, the VCG mechanics played out in exactly the same way.[20] In both assignment areas, Dish Network won one

[20]The assignment of PEAs 205 and 213 (Douglas City, CA and Bend, OR) occurred in assignment round 29 (REAG 6) and the assignment of PEAs 119, 206 and 297 (Yakima, WA; Wenatchee, WA and Pendleton, OR) occurred in assignment round 37 (REAG 6).

generic block, while T-Mobile and U.S. Cellular collectively won 6 generic blocks; and in both areas, an indirect bidder complementarity between two opponents prevented Dish Network from acquiring block G, creating core violations of \$101,030 and \$147,602, respectively. Most conspicuously, an entrant was relegated to the A block (which, as previously argued, appears to have been the least desirable block), at zero cost to the two incumbents.

Our second exercise is to determine which assignment areas had VCG outcomes lying outside the core. We find 38 such instances (including the three zero-revenue instances identified above). The cumulative revenue shortfall across these 38 assignment areas was \$4,411,699. That is, if the FCC had instead used a core-selecting payment rule—and assuming the same bids—the assignment stage gross revenues would have been \$140,342,331 rather than \$135,930,632 (about 3.25% higher). However, given that core-selecting auctions are not in general incentive compatible (i.e., bidders may gain from non-truthful bidding), the interpretation of this comparison is limited.

We then perform three additional exercises to identify other assignment areas in which the bids had the "potential" for non-core outcomes—or presented bidder coalitions with opportunities for egregious manipulations utilizing bidder complementarities.

To motivate our third exercise, recall that the presence of complementarities is necessary, but not sufficient, to generate core violations. When all bidders have substitutes preferences, changing the magnitude of bidders' relative preferences by rescaling their bids would never produce a bidder complementarity, and consequently would never trigger a core violation. However, when valuations already contain complementarities, rescaling the valuations may cause complementarities that were irrelevant at the optimal allocation to become relevant. Therefore, rescaling bids is an easy way to show that an assignment area had the "potential" for non-core outcomes (without artificially introducing any new bidder complementarities). Our third exercise looks for assignment areas where rescaling the actual bids would give rise to core violations.

The fourth and fifth exercises are motivated by Ausubel and Milgrom (2002), who first described an unusual vulnerability of the VCG mechanism dubbed "loser collusion". In the VCG mechanism, any individual bidder is incentivized to bid truthfully, but coalitions of bidders can profit from certain group deviations. For starters, a coalition can usually gain by eliminating opportunity costs imposed by members of the coalition on each other. To eliminate such costs in the assignment stage, members of a coalition can bid zero for each assignment option that they do not win (i.e., by reducing their losing bids). There is nothing unorthodox about this form of collusion, so we refer to any accrued coalitional gains from reducing their losing bids as *normal collusive gains*.

However, in a VCG assignment stage, a coalition of bidders can sometimes further gain by increasing its members' bids on assignment options that they win (i.e., by raising their winning bids). They find this to be beneficial when coalition members are complementary to each other and their group deviation exploits a core violation. Any additional gains obtained from raising the winning bids, after setting all losing bids to zero, will be referred to as *excess collusive gains*.[21] Note that: (1) reducing losing bids and raising the winning bid never affects the bidder's own payoff; and (2) by Proposition 4, a coalition C can never earn excess collusive gains when the coalitional value function of its opponents in $N \setminus C$ is assignment submodular.

A coalition of bidders is said to have a *winner collusion opportunity* when it earns excess collusive gains from a group deviation that does not alter the winning assignment. A coalition is said to have a *loser collusion opportunity* when a group deviation alters the winning assignment while simultaneously generating excess collusive gains that exceed the gains from the winner collusion opportunity (when available to the coalition). The key difference between the two collusion concepts is that the latter leads to inefficiency. Finally, a *pure loser collusion opportunity* is a loser collusion opportunity for bidders who get their least desirable options in the efficient assignment (this corresponds to the original definition of "loser collusion" from Ausubel and Milgrom (2002)).

In the fourth exercise, we look for coalitions with winner collusion opportunities, and in the fifth exercise, we look for loser collusion opportunities.[22] Observe that we treat bidders' bids as their true values here. On the one hand, this is an easy assumption to make, given the dominant strategy property of the VCG mechanism for individual bidders. On the other hand, we are checking for group deviations from misreporting true values. Such group deviations can be quite lucrative within the sequential assignment stage process of the Incentive Auction, where the same bidders bid repeatedly against each other. At the same time, these deviations may be viewed as individually risky, in that each coalition member raises its own bid and risks paying more only to improve the payoff of other coalition members (whose identities are anonymous). As such, the assumption that bids correspond to true values may be viewed as reasonable.

An interested reader can find the complete methodological description for all of the empirical exercises and a detailed illustrative example (using the bidding data from the assignment round in Milwaukee, WI) in the appendix.

Our empirical results are summarized in Table 5. Note that we can exclude 21 of the 228 assignment areas from consideration, as it is never possible to observe a non-core VCG outcome or an affirmative

[21] It is important to define excess collusive gains as a payoff increase only after all losing bids are set to zero. When the losing bids are not set to zero, an increase in the winning bid can be equivalent to a reduction in losing bids, thus reflecting normal gains rather than excess gains.

[22] To eliminate any normal collusive gains, we first reduce all losing bids to zero for all members of a coalition. For winner collusion, we set coalition members' winning bids to the minimum levels required for winning and check whether the members' payoffs increase when they simultaneously raise their winning bids from this point. For loser collusion, we determine whether coalition members can increase their payoffs even more, compared to their optimal payoff under winner collusion, by strategically raising some of their losing bids and thereby altering the winning assignment.

answer in any of our exercises, in an assignment round with only two bidders.[23] Thus, we treat the sample size as being the 207 assignment areas with three or more bidders.

Table 5: *Summary of Empirical Exercises for the Incentive Auction Bidding Data*

	Exercise 1	*Exercise 2*	*Exercise 3*	*Exercise 4*	*Exercise 5*	
	Zero-Revenue Non-Core Outcomes	Non-Core Outcomes	Rescaling Values	Winner Collusion	Loser Collusion	Pure Loser Collusion
Number of Markets	18	207	207	207	207	55
Identified Markets	3	38	103	149	110	37
Number of Coalitions	–	–	–	1626	1626	55
Identified Coalitions	–	–	–	349	170	37
Classification of Underlying Bidder Complementarities						
Direct	0	7 (18.5%)	–	58 (17%)	67 (39%)	7 (19%)
Indirect	3 (100%)	29 (76.5%)	–	291 (83%)	103 (61%)	30 (81%)
Mixed	0	2 (5%)	–	–	–	–

In addition to identifying non-core outcomes, we classify the underlying bidder complementarity that causes the non-core outcome or that is being exploited by a colluding coalition. "Direct" and "Indirect" instances correspond to direct and indirect bidder complementarities, and "Mixed" category contains instances in which there are multiple core violations, some of them caused by direct bidder complementarities and some of them by indirect complementarities.

Starting from the 38 original assignment areas with non-core outcomes, the rescaling exercise finds an additional 65 areas with bidder complementarities that could have potentially caused non-core outcomes. The winner collusion exercise identifies 149 assignment areas in which coalitions could have exploited bidder complementarities for excess gains. Lastly, the loser collusion exercise finds 110 areas in which coalitions could have gained even more (compared to their winner collusion opportunities) by forcing a different assignment. In particular, for pure loser collusion, we consider all coalitions with zero VCG payoffs (there are 55 of them in total, each one in a different assignment area) and we find that in 37 of them losers could gain by altering the assignment. Cumulatively, these exercises reveal 134 additional assignment areas with bidder complementarities, bringing the total number of areas with potential issues to 172 (out of 207). Our classification of underlying complementarities further reveals that indirect bidder complementarities—the ones arising from enforced contiguity—appear to play the dominant role in driving these results.

[23] With two bidders, it is well known that the VCG outcome lies within the core for all possible valuations. Similarly, it is easy to see that profitable winner collusion or loser collusion cannot occur in auctions with only two bidders.

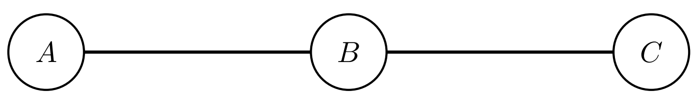
These empirical results must be taken with some degree of caution since the Broadcast Incentive Auction data is limited to markets in which the number of generic blocks is seven, a prime number that rules out any possibility of symmetries that could have reduced the number of problematic instances. Nevertheless, these results suggest that the presence of various bidder complementarities, both direct and indirect, is a systematic feature of asymmetric assignment stages rather than something rare and exotic.

8 Complementarities Arising from Other Feasibility Constraints

In the previous sections, we have found that a key source of the intensified bidder complementarities that arise in assignment stages is the contiguity requirement typically imposed by spectrum regulators. The purpose of the current section is to show that a contiguity requirement is not the only type of feasibility constraint that creates or amplifies bidder complementarities—in principle, any complex feasibility constraint can have a similar effect.

To illustrate, consider the scenario depicted in Figure 1. An auctioneer wishes to allocate three possible items (A, B and C), subject to the following restrictions: allocating A conflicts with allocating B; allocating B conflicts with allocating C; but allocating A does not conflict with allocating C. As such, conflicts among allocating these items are not transitive. Moreover, the bidders seeking the non-conflicting pair of items prove to be complementary.

Figure 1: *Non-Transitive Conflicts*



Conflicts having this character are not uncommon. They occur in the setting of the Broadcast Incentive Auction's reverse auction. Three broadcasters seek television licenses for the same channel at locations approximately along a straight line. The broadcast locations for A and B are 60 miles apart, and similarly the broadcast locations for B and C are 60 miles apart—sufficiently close that their signals would create interference with each other. However, the broadcast locations for A and C are 120 miles apart—sufficiently distant that television stations A and C can coexist. Indeed, this reasoning shows that the complex feasibility constraint in the reverse auction of the Broadcast Incentive Auction gives rise to bidder complementarities.

Similar phenomena can also arise completely outside the radio spectrum area. Suppose that a franchisor wishes to allocate franchises by auction. At the same time, franchisees may require some degree of geographic exclusivity for their franchises to be profitable (e.g., the franchisor may need to guarantee to each franchisee that no other franchise will be opened within a 60-mile radius). The feasibility constraint in the franchise auction would then operate in exactly the same way as in the television license auction.

In a real-world example, ICANN (the Internet Corporation for Assigned Names and Numbers) began accepting applications in 2012 for new generic top-level domains. Unique qualifying applications for different domain names would be granted, but ICANN's policy was to avoid granting applications for names that would be confusingly similar to one another. For example, the evaluators might decide that the domains *.car* and *.cars* are too close to each other and could not both be issued. They might also decide that the domains *.cars* and *.cares* are too close to each other and could not both be issued. However, the evaluators might feel that the domains *.car* and *.cares* are sufficiently far apart that they can both coexist.

Particular implementations of competition policy in spectrum auctions can also produce non-transitive conflicts. Consider an auctioneer seeking to allocate two licenses ($East$ and $West$) among three bidders: incumbent A, who is interested only in $East$; incumbent B, who is interested only in $West$; and entrant C. Suppose that the auctioneer utilizes a "floating" set-aside, in which one of the two licenses is reserved for an entrant, but it is not pre-specified which license is reserved. (Often a floating set-aside is equivalent to an aggregate cap on what the incumbents as a group are allowed to win, in this case, one license.) If entrant C is only interested in $West$, then the conflict depicted in Figure 1 emerges from this scenario: the common interest in the $West$ license creates a conflict between B and C, while the floating set-aside creates a conflict between A and B, but there is no conflict between A and C.[24]

Observe that, in any scenario captured by Figure 1, the bidders for A and C are perfect complements. Indeed, the scenario creates exactly the same complementarities as in the standard LLG model (where the bidder for B in Figure 1 takes the role of the "global" bidder and the bidders for A and C take the roles of "local bidders"). More complex graphs of conflict interactions will create more complex patterns of complementarities. Similar effects may also occur in other environments with complex feasibility constraints, such as those present in electricity grids or when allocating airport landing/takeoff slots. Following the same reasoning as in previous sections, such settings can generate anomalies in the VCG mechanism. Zero-revenue and other non-core outcomes may arise. Winner collusion and even loser collusion become possible. The results may look eerily similar to those of assignment stages, even though there is no contiguity requirement imposed.

9 Conclusion

The possibility of a non-core, zero-revenue outcome of the VCG mechanism has been known for 20 years. In this article, we demonstrate that this anomaly is not merely a theoretical curiosity but

[24] At the same time, observe that certain other implementations of competition policy will never by themselves create bidder complementarities, for example, conventional "spectrum caps" (i.e., individual limits on the number of licenses that bidders can win) and fixed set-asides (reservations of particular licenses for entrants).

a real problem, as we document its occurrence in the field for the first time: in the high-profile, high-stakes FCC Broadcast Incentive Auction.

Knowing the theory that we have developed herein, it is not at all surprising that zero-revenue outcomes manifested themselves in the assignment stage of a spectrum auction. We demonstrated that complementarities among bidders can arise *directly* or *indirectly*. While the former type of complementarity is derived from complementarities in goods, the latter type is triggered by the joint restriction on the outcome of the assignment stage (all winners must receive contiguous assignments). When either type of complementarity comes into play in the VCG mechanism, complementary bidders are not individually responsible for depriving their opponent of its choice—and so they escape bearing the full opportunity cost of depriving their opponent of its choice.

Just because bidder complementarities (and non-core outcomes) are exacerbated by assignment stages in theory and just because non-core outcomes are observed there in practice does not imply that the two-stage approach should be abandoned. Quite the opposite, it has numerous practical advantages, including the mitigation of anti-competitive strategies and the simplification of bidding. For what it is worth, the spectrum regulators of the US, UK, Canada and Australia have adopted allocation stages for generic blocks, followed by assignment stages, in most of their recent auctions. But our findings, that bidder complementarities both should be expected in assignment stages and actually do empirically occur there, highlight the practical limitations of the VCG mechanism. One needs to be cautious about utilizing VCG for what would otherwise seem to be a natural application, knowing that revenues may become uncompetitively low and that the mechanism may become susceptible to loser collusion and a laundry list of other vulnerabilities.

What about alternatives to the VCG mechanism, including the first-price menu auction (which is itself a core-selecting mechanism) and various minimum-revenue core-selecting auctions (e.g., the "nearest-VCG" mechanism)? Our results indicate the prevalence of bidder complementarities in assignment stages. From first principles, we know that a complementarity among costly actions of agents trying to achieve a common goal must give rise to a free-rider problem for some agents. In the VCG mechanism, the full burden of the free-rider problem is borne by the auctioneer, who accepts the uncompetitively low revenues of a non-core outcome. Switching to a core-selecting mechanism simply pushes the free-rider problem onto the complementary bidders, who will respond by bidding conservatively in an attempt to free-ride on one another, and the auctioneer would still face some degree of depressed revenues.[25] To summarize, the systematic presence of bidder complementarities generally leads to uncompetitively low revenues, irrespective of the payment rule.

But even if all auction formats suffer from low revenues, one should not prematurely conclude that the VCG mechanism is the right choice for an assignment stage. It does create the casual appearance of offering superior efficiency properties relative to core-selecting auctions (which are generally not

[25] Ausubel and Baranov (2020).

incentive compatible), but only if the VCG mechanism is taken at face value. However, as we have seen, low revenue is only one of several disturbing pathologies of the VCG mechanism under bidder complementarities. The possibility of non-core outcomes also opens the door to winner collusion, loser collusion, and the low-cost denial of preferred assignments. Given these anomalies, any expectation that the VCG mechanism would elicit truthful values or achieve efficiency in the wider sense is probably naïve. Making matters worse, assignment stages in spectrum auctions with regional licenses are routinely run sequentially (one region after another), with the same bidders participating repeatedly and having opportunities to learn how to exploit these vulnerabilities.

Thus, it appears that there are advantages and disadvantages both to the VCG mechanism and to alternative auction formats for assigning frequencies. In practice, all assignment stages of which we are aware, prior to the Broadcast Incentive Auction, used a core-selecting mechanism (typically a weighted or unweighted "nearest-VCG" mechanism). The reader may wonder why the FCC broke with precedent and instead adopted a VCG mechanism for the assignment stage. The explanation is rather auction-specific. The amount of spectrum to be repurposed in the Broadcast Incentive Auction would be endogenously determined and, in order for a given quantity of spectrum to be cleared, the forward auction's revenues would be required to exceed the reverse auction's clearing costs. This balanced-budget constraint would be based solely on the first, allocation stage and would not include revenues from the second, assignment stage (which had not yet occurred). There were concerns that assignment stage payments might divert revenues away from the allocation stage, reducing the probability of meeting the balanced-budget constraint. As a result, the lowest coherent payment rule was adopted for the assignment stage.

We conclude this article with three observations: First, as of this writing, the FCC has conducted four more spectrum auctions taking the two-stage approach of an allocation stage for generic spectrum followed by an assignment stage for assigning specific frequencies. Even though one of these auctions also had a revenue requirement with the potential to be binding, the FCC chose to utilize a core-selecting mechanism (rather than a VCG mechanism) for the assignment stage of all four of these auctions.

Second, while a preference for contiguity in frequency assignments is prevalent in spectrum auctions, similar value structures may arise naturally in other settings where other dimensions such as time may take on the role that frequency has taken here. For example, many generators in electricity auctions have preferences for contiguous dispatch times, in order to minimize ramp-up and ramp-down costs. Similarly, when government securities and other financial products transact in an auction, different buyers and sellers may wish to borrow or lend in different time periods, but most participants are likely to value contiguous time periods. As in this article, contiguity restrictions will give rise to bidder complementarities, creating similar anomalies and raising similar concerns.

Third, while the indirect complementarities encountered in this article were driven by a particular contiguity restriction, we have seen in Section 8 that similar phenomena can occur whenever a complex feasibility constraint is applied. As such, the clear message of this article that widespread complementarities may prevent the VCG mechanism from being a panacea can perhaps be taken as a more universal warning to mechanism designers.

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A Appendix - Proofs

PROOF OF PROPOSITION 2. Since $v(\cdot)$ satisfies GS, it is $M^\natural$-concave. Then, by Corollary 1.3 in Murota and Shioura (2018), for each $q = 0, ..., m$, function $v^q(\cdot)$ defined as

$$v^q(z) \;=\; \begin{cases} v(z) & z \in \Omega(q) \\ -\infty & z \notin \Omega(q) \end{cases} \tag{A.1}$$

is also $M^\natural$-concave (satisfies GS). Note that

$$\begin{aligned} x(p|q) \;&=\; \arg\max_{z\in\Omega(q+)} \{v(z|q) - p\cdot z\} \;=\; \arg\max_{z\in\Omega(q)} \{v(z|q) - p\cdot z\} \\ &=\; \arg\max_{z\in\Omega(q)} \{v(z) - p\cdot z\} \;=\; \arg\max_{z\in\Omega} \{v^q(z) - p\cdot z\} \end{aligned} \tag{A.2}$$

Since demand correspondence $x(p|q)$ is the same for $v(\cdot|q)$ and $v^q(\cdot)$, $v(\cdot|q)$ satisfies GS on $\Omega(q+)$ and $v(.)$ satisfies assignment substitutes for each quantity q. *QED.*

An alternative proof that assignment substitutes for q imply assignment submodularity for q: For any bundle $z \in \Omega(q+)$, denote $x^q(z)$ the most preferred assignment option from $\Omega(z|q)$, i.e, $v(z|q) = v(x^q(z)|q)$. Suppose that $v(\cdot|q)$ satisfies assignment substitutes ($M^\natural$-concavity) for quantity q, but it is not assignment submodular for the same quantity q (then it must be that $q \geq 2$). Then there exist bundles $z, z' \in \Omega(q+)$ such that $z \subset z'$ and item $k \in M \setminus z'$ such that

$$v(z+k|q) \; - \; v(z|q) \; < \; v(z'+k|q) \; - \; v(z'|q) \; . \tag{A.3}$$

Note that for (A.3) to hold, it must be that $v(z'+k|q) > v(z'|q)$ and as a result, $k \in x^q(z'+k)$. By $M^\natural$-concavity for quantity q, there must exist item $h \in x^q(z) \setminus x^q(z'+k)$ such that

$$\begin{aligned}
v(x^q(z'+k)|q) \; + \; v(x^q(z)|q) \; &\leq \; v(x^q(z'+k)-k+h|q) \; + \; v(x^q(z)+k-h|q) && M^\natural\text{-concavity} \\
&\leq \; v(z'|q) \; + \; v(z+k|q) && \textit{by construction} \\
&< \; v(z'+k|q) \; + \; v(z|q) && \textit{due to } (A.3) \\
&= \; v(x^q(z'+k)|q) \; + \; v(x^q(z)|q)
\end{aligned}$$

which is a contradiction. □

PROOF OF PROPOSITION 4. The Vickrey payments never exceed the upper bound on core payments. To verify the lower bound, consider coalition $C \subset N$ and its complement $\widetilde{C} = N \setminus C$. For each bidder $j \in \widetilde{C}$, we have

$$p_j^V \; = \; w_{N\setminus j}(X) \; - \; w_{N\setminus j}(X^e_{N\setminus j}) \; \geq \; w_C(X^e_{\widetilde{C}\setminus j}) \; - \; w_C(X^e_{\widetilde{C}}) \tag{A.4}$$

Without loss of generality, suppose that $\widetilde{C} = \{1, 2, ..., l\}$ where $l = |\widetilde{C}|$. Note that submodularity of $\widetilde{w}_C(\cdot)$ implies that

$$w_C(X^e_{\widetilde{C}\setminus j}) \; - \; w_C(X^e_{\widetilde{C}}) \; \geq \; w_C(X^e_{\widetilde{C}\setminus(j+k)}) \; - \; w_C(X^e_{\widetilde{C}\setminus k}) \qquad \forall j, k \in \widetilde{C} \tag{A.5}$$

Intuitively, the fewer bidders from $\widetilde{C}$ assigned their efficient assignments, the larger the selection of blocks for bidders in C. Due to submodularity, the marginal effect of extra blocks must be nonincreasing. We have:

$$\begin{aligned}
p_1^V \; &\geq \; w_C(X^e_{\widetilde{C}\setminus 1}) \; - \; w_C(X^e_{\widetilde{C}}) \; &&= \; w_C(X^e_{\widetilde{C}\setminus 1}) \; - \; w_C(X^e_{\widetilde{C}}) \\
p_2^V \; &\geq \; w_C(X^e_{\widetilde{C}\setminus 2}) \; - \; w_C(X^e_{\widetilde{C}}) \; &&\geq \; w_C(X^e_{\widetilde{C}\setminus(1+2)}) \; - \; w_C(X^e_{\widetilde{C}\setminus 1}) \\
p_3^V \; &\geq \; w_C(X^e_{\widetilde{C}\setminus 3}) \; - \; w_C(X^e_{\widetilde{C}}) \; &&\geq \; w_C(X^e_{\widetilde{C}\setminus(1+2+3)}) \; - \; w_C(X^e_{\widetilde{C}\setminus(1+2)}) \\
&\dots \\
p_l^V \; &\geq \; w_C(X^e_{\widetilde{C}\setminus l}) \; - \; w_C(X^e_{\widetilde{C}}) \; &&\geq \; w_C(X) \; - \; w_C(X^e_{\widetilde{C}\setminus(1+\ldots+l-1)})
\end{aligned}$$

By summing the above inequalities, the lower bound is established:

$$\sum_{j\in N\setminus C} p_j^V \; \geq \; w_C(X) \; - \; w_C(X^e_{\widetilde{C}}) \; . \tag{A.6}$$

□

PROOF OF PROPOSITION 5. Consider coalition $C \subset N$ such that $i \in C$ and its complement $\widetilde{C} = N \setminus C$. For each bidder $j \in \widetilde{C}$, we have

$$p_j^V \;=\; w_{N\setminus j}(X) \;-\; w_{N\setminus j}(X^e_{N\setminus j}) \;\geq\; w_i(X^e_{N\setminus(j+i)}) \;-\; w_i(X^e_{N\setminus i}) \tag{A.7}$$

Without loss of generality, suppose that $\widetilde{C} = \{1, 2, ..., l\}$ where $l = |\widetilde{C}|$. Note that submodularity of $v_i(\cdot)$ implies that

$$w_i(X^e_{N\setminus(i+j)}) \;-\; w_i(X^e_{N\setminus i}) \;\geq\; w_i(X^e_{N\setminus(j+k+i)}) \;-\; w_i(X^e_{N\setminus(i+k)}) \qquad \forall j,k \in \widetilde{C}$$

Intuitively, the fewer bidders from $\widetilde{C}$ assigned their efficient assignments, the larger the selection of blocks for bidders in C. Due to submodularity, the marginal effect of extra blocks must be nonincreasing. We have:

$$\begin{array}{lllll}
p_1^V & \geq & w_i(X^e_{N\setminus(1+i)}) \;-\; w_i(X^e_{N\setminus i}) & = & w_i(X^e_{N\setminus(1+i)}) \;-\; w_i(X^e_i) \\
p_2^V & \geq & w_i(X^e_{N\setminus(2+i)}) \;-\; w_i(X^e_{N\setminus i}) & \geq & w_i(X^e_{N\setminus(1+2+i)}) \;-\; w_i(X^e_{N\setminus(1+i)}) \\
p_3^V & \geq & w_i(X^e_{N\setminus(3+i)}) \;-\; w_i(X^e_{N\setminus i}) & \geq & w_i(X^e_{N\setminus(1+2+3+i)}) \;-\; w_i(X^e_{N\setminus(1+2+i)}) \\
... & & & & \\
p_l^V & \geq & w_i(X^e_{N\setminus(l+i)}) \;-\; w_i(X^e_{N\setminus i}) & \geq & w_i(X^e_{C\setminus i}) \;-\; w_i(X^e_{N\setminus(1+2+...+(l-1)+i)})
\end{array}$$

By summing the above inequalities, the lower bound is established:

$$\sum_{j\in N\setminus C} p_j^V \;\geq\; w_i(X^e_{C\setminus i}) \;-\; w_i(X^e_i)\;. \tag{A.8}$$

□

PROOF OF PROPOSITION 6. If a bidder cannot be assigned the favored (or poisoned) block due to the contiguity restriction, the truthful bids of this bidder have no impact on the VCG outcome of the assignment stage. Without loss of generality we can assume that the favored (or poisoned) block may be assigned to any bidder in $N(y)$ participating in the assignment stage, notwithstanding the contiguity restriction.

For a setting with a single favored block, the VCG assignment stage is equivalent to a standard second-price auction in which $n(y)$ bidders compete for a single item (an assignment with the favored block). The bidder with the highest value for the favored block (the winner) gets an assignment with the favored block and pays the second-highest value. It is well known that the outcome of the second-price auction is in the core.

For a setting with a single poisoned block, the VCG assignment stage is equivalent to a standard uniform-price auction, where $n(y)$ bidders compete for $n(y) - 1$ items (assignments without the poisoned block). The bidder with the lowest disutility from the poisoned block (the loser) gets the poisoned block, while its bid sets the price for all other bidders with greater disutility (winners) who avoid assignments with the poisoned block. It is well known that the outcome of the uniform-price auction among bidders with unit demands is in the core. □

PROOF OF PROPOSITION 8. Given the asymmetry, there must be a pair of bidders i and j such that $y_i < y_j$. Suppose that both i and j bid $\alpha > 0$ for their bottom assignment option (blocks $1, ..., y_i$ for bidder i and $1, ..., y_j$ for bidder j) and zero on all other feasible options. Also suppose that a different bidder k bids α for the assignment option corresponding to blocks $y_i + 1, ..., y_i + y_k$ (which is feasible for bidder k) and zero otherwise, and that all other bidders (if any) bid zero for all of their options. The resulting VCG outcome is outside the core since bidders i and k win assignments $1, ..., y_i$ and $y_i + 1, ..., y_i + y_k$, respectively, while paying zero (by the same logic as in Example 1 from Section 3). Finally, by our observation in Section 6, any set of bids submitted in the assignment stage with the contiguity restriction can always be represented by an additive value function. □

PROOF OF PROPOSITION 9. Consider an asymmetric assignment stage y in which bidder i has strict assignment preferences. Denote $\bar{z}$ bidder i's most preferred feasible assignment option in assignment stage y (i.e., unique bundle $z \in \Omega(y_i)$ such that $v_i(z|y_i) = w_i(X)$). We say that the members of coalition $N(y) \setminus i$ bid for assignment (bandplan) $x = (x_1, ..., x_n) \in X$ if their reported values are given by

$$v_j(z|y_j) \;=\; \begin{cases} \alpha & z = x_j \\ 0 & z \neq x_j \end{cases} \qquad \forall j \in N(y) \setminus i \;,$$

where $\alpha > 0$. Note that when coalition $N(y) \setminus i$ bids for assignment x, their bids do not overlap with each other. With α being sufficiently large, the efficient assignment $x^e = x$. Note that members of coalition $N(y) \setminus i$ do not impose any opportunity costs on each other or bidder i. As a result, bidder i gets x_i and pays 0 (so bidder i is forced to x_i). The payments of coalition members depend on values of bidder i only. We consider two cases: $n(y) = 3$ and $n(y) \geq 4$.

$n(y) = 3$: There are three bidders (i, j, k) bidding for assignments corresponding to y_i, y_j, y_k blocks. Bidder i has four distinct positions (three if $y_j = y_k$) in the band: two end positions (at the bottom, or at the top of the band) and two middle positions (only one if $y_j = y_k$). *End positions:* WLOG, suppose that $y_j \neq y_i$, and bidder i's most preferred bandplan is $(i, ., .)$ (i.e., $\bar{z}$ corresponds to y_i lowest blocks). Suppose that coalition bids for the (j, k, i) bandplan which triggers a direct bidder complementarity when $y_j < y_i$ or an indirect bidder complementarity when $y_j > y_i$. Vickrey payments are given by $p_j^V = 0$ and $p_k^V = w_i(X_j^e) - w_i(X_i^e)$. Then, there exists a detectable core violation for bidder i and coalition $C = \{i\}$ because

$$p_j^V + p_k^V \;=\; w_i(X_j^e) \;-\; w_i(x_i^e) \;<\; w_i(X) \;-\; w_i(X_i^e) \;=\; w_i(X_{C\setminus i}^e) \;-\; w_i(X_i^e) \;.$$

Middle positions: WLOG, suppose that $y_j \neq y_k$ and bidder i's most preferred bandplan is (j, i, k). If $y_j < y_k$, then coalition bids for (j, k, i) bandplan; if $y_j > y_k$, then coalition bids for (k, j, i) bandplan. Both scenarios trigger a direct bidder complementarity with one bidder (j or k) paying zero and the other bidder paying less than $w_i(X) \;-\; w_i(X_i^e)$, leading to a detectable core violation for bidder i and coalition $C = \{i\}$. Finally, if $y_j = y_k$, bidder i has only one middle position. In this case, it is impossible for the coalition to trigger a bidder complementarity (direct or indirect), so (5.12) is satisfied for any bids of the coalition.

$n(y) \geq 4$: *(The high level idea of the proof here is to show that there is always a bandplan that is consistent with option $\bar{z}$ such that there is an asymmetric 3-bidder subband that includes option $\bar{z}$ for bidder i as an end position, or one of the two middle positions. Then the logic of $n(y) = 3$ case can be applied to set up a detectable core violation for bidder i and coalition that includes all other bidders with the exception of two bidders from the 3-bidder subband.*

Option $\bar{z}$ for bidder i must be consistent with some bandplan $(j_1, j_2, ..., j_{l-1}, i, j_{l+1}, ..., j_{N(y)-1})$ where bidders $j_1, ..., j_{l-1}$ are assigned lower blocks than bidder i, and bidders $j_{l+1}, ..., j_{N(y)-1}$ are assigned higher blocks. Note that $\bar{z}$ is also consistent with alternative bandplans in which bidders $j_1, ..., j_{l-1}$ are assigned the same lower blocks, but in a different order, and bidders in $j_{l+1}, ..., j_{N(y)-1}$ are assigned the same higher blocks, but in a different order. With $n(y) \geq 4$, at least one three-bidder subband exists, either (j_{l-2}, j_{l-1}, i) or (i, j_{l+1}, j_{l+2}), where bidder i occupies the end position. If at least one of them is asymmetric, the logic of $n(y) = 3$ case applies. If both subbands are symmetric, there must be bidder j such that $y_j \neq y_i$ and an alternative bandplan consistent with $\bar{z}$ in which bidder j is situated adjacent to bidder i (directly below or above), resulting in an asymmetric three-bidder subband with $\bar{z}$ corresponding to the end position. The only exception is a bandplan in which bidder j gets the end position, bidder i is adjacent to bidder j, i.e., $j, i, j_2, ..., j_{N(y)-1}$, and all bidders $j_2, ..., j_{N(y)-1}$ won the same number of generic blocks as bidder i. Then, the logic of the $n(y) = 3$ case can be applied to the middle position of the (j, i, j_2) subband since bidder j has won a different number of generic blocks than bidder j_2. As a result, for any $\bar{z}$, we can construct a bandplan such that two coalition bidders (denote them as j and k) prevent bidder i from winning option $\bar{z}$ while causing a bidder complementarity. When all bidders in $N(y) \setminus i$ bid for the identified bandplan, there will be a detectable core violation for bidder i and coalition $C = N(y) \setminus (j + k)$.

□

B Appendix - An Illustrative Example for the Empirical Exercise

The methodological description of each empirical exercise is as follows:

- *Exercise 1:* For each assignment area with zero revenues, does the VCG outcome calculated from the actual bids lie outside of the core?
- *Exercise 2:* For each assignment area, does the VCG outcome calculated from the actual bids lie outside of the core?
- *Exercise 3:* For each assignment area, does the VCG outcome calculated from any rescaled version of the bids lie outside of the core? To address this, we apply all possible combinations of scaling factors chosen from the nine-element set $\{\, \epsilon + 0.25k : \; k = 0, 1, ..., 8 \,\}$ independently

to the bids of all of the bidders (i.e., we rescale each bid in the range from essentially 0% to 200% in 25% increments), and we recalculate the VCG outcome for each such combination.

- *Exercise 4:* For each assignment area, is there a bidder coalition such that winner collusion is profitable? We consider all possible coalitions of size two and above (excluding the grand coalition)—and we check whether coalition members can increase their payoffs by changing their winning bids (after setting their losing bids to zero).

- *Exercise 5:* For each assignment area, is there a bidder coalition such that loser collusion is profitable? We consider all possible coalitions of size two and above (excluding the grand coalition)—and we check whether coalition members can increase their payoffs even more, compared to their optimal payoffs under the fourth exercise, by strategically raising some of their losing bids. For "pure" loser collusion, we only consider coalitions with zero VCG payoffs.

We illustrate exercises 3 – 5 using the bidding data from the assignment round 21, in which seven blocks were assigned in PEA 38 (Milwaukee, WI) and the VCG outcome was in the core. All bids submitted by bidders in this round are reported in Table 6, with the winning bids displayed in bold.

Table 6: *Assignment Bids for PEA 38 (Milwaukee, WI) in $*

Bandplan

A	B	C	D	E	F	G

Bidders (generic winnings)	Dish Network (1 block)	T-Mobile (3 blocks)	U.S. Cellular (2 blocks)	New Level (1 block)
Compatible Bids (block(s), bid)	A, 0	ABC, 0	AB, 10.3k	**A, 0**
	B, 0	**BCD, 20.2m**	BC, 0	B, 176.471k
	C, 0	CDE, 13m	CD, 50.2k	C, 176.471k
	D, 0	DEF, 1m	DE, 50.2k	D, 176.471k
	E, 0	EFG, 0.1m	**EF, 0**	E, 176.471k
	F, 1019.752k		FG, 10.3k	F, 176.471k
	G, 250.123k			G, 0
VCG payments	**186.771k**	**819.830k**	**0**	**0**

To illustrate the rescaling exercise (Exercise 3), observe that given T-Mobile's very high bid for the BCD assignment, the rest of the assignment is determined by comparing Dish Network's bid for block G, on the one hand, with the sum of New Level's and U.S. Cellular's bids for the EFG blocks, on the other. This is tantamount to the "LLG" model with indirect bidder complementarities, where Dish Network takes the role of the global bidder, while New Level and U.S. Cellular take the role of local bidders. With the original bids, the global bidder prevails over the two local bidders, since $250.123k > 186.771k = 176.471k + 10.3k$, and so the VCG outcome lies within the core. However, had the global bidder submitted sufficiently smaller bids–or had the local bidders submitted sufficiently larger bids—a core violation would have been triggered instead. Rescaling the bids and recomputing the VCG outcomes would detect such a scenario.

To illustrate our winner collusion exercise (Exercise 4), consider a coalition consisting of Dish Network and U.S. Cellular. The opportunity costs of Dish Network winning block G come from U.S. Cellular (a cost of $10,300 from winning EF instead of FG) and New Level (a cost of $176,471 from winning A instead of E). Obviously, U.S. Cellular can eliminate its $10,300 in opportunity costs imposed on Dish by bidding zero on FG ("normal" collusive gains). Now suppose that both Dish Network and U.S. Cellular further increase their winning bids. Observe that Dish Network's bid for block G and U.S. Cellular's bid for EF create an indirect bidder complementarity, since each one by itself prevents New Level from winning block E. As a result, Dish Network's opportunity costs are directly reduced by the amount of U.S. Cellular's winning bid. By raising its bid on EF, U.S. Cellular can reduce Dish Network's VCG payment all the way to zero, for a total coalitional gain of $176,471 ("excess" collusive gains). Our winner collusion exercise considers which coalitions are able to extract such excess collusive gains.

Finally, to illustrate our loser collusion exercise (Exercise 5), continue considering the coalition consisting of Dish Network and U.S. Cellular and suppose that, instead of the winner-collusion deviations, Dish Network raises its bid for F and U.S. Cellular raises its bid for AB (both losing options). With sufficiently high bids, the coalition will be successful in shifting T-Mobile from the BCD assignment to the CDE assignment. Observe that this manipulation imposes $7.2m in opportunity costs due to T-Mobile. However, since both bids conflict with T-Mobile's bid for BCD, creating an indirect bidder complementarity, neither Dish Network nor U.S. Cellular is individually responsible for displacing T-Mobile. As a result, the coalition gains $790,230 ($779,930 for Dish Network and $10,300 for U.S. Cellular) from loser collusion, compared to only $186,771 ($176,471 + $10,300) from the winner collusion manipulation described above. Most importantly, observe that this manipulation causes a value loss of $6.42m to this market—the substantial collateral damage caused by loser collusion.

For an example of pure loser collusion, consider the coalition consisting of U.S. Cellular and New Level, each winning its least desirable option and obtaining a zero payoff. If their bids for options FG and E, respectively, were made substantially higher, they would have outbid Dish Network's bid for G and, due to their indirect bidder complementarity, would have paid zero and gained $186,771 in total. Our loser collusion exercise determines which coalitions are able to extract additional collusive gains by altering the winning assignment.